Problems 25-28

25. Define the first moment of the four momentum of a distribution of matter in flat spacetime as

$$V^{\alpha\beta} = \int x^{\alpha} T^{0\beta} d^3\vec{x}$$

(a) Show that the time derivative of *V* is $\partial_0 V^{\alpha\beta} = \int T^{\alpha\beta} d^3\vec{x}$. It may be helpful to consider separately the cases $\alpha = 0$ and $\alpha = i$.

(b) From this, show that the angular momentum as defined in class is conserved. Also, show that the dipole moment satisfies $\vec{D}(t) = \vec{D}(0) + \vec{P}t$.

26. Show that for a stationary distribution of matter ($\partial_0 T^{\alpha\beta} = 0$),

(a) $\int T^{ij} d^3\vec{x} = 0$

(b) $\int x^{(i} T^{j)k} d^3\vec{x} = 0$ (hint: consider the time derivative of $\int x^i x^j T^{0k} d^3\vec{x}$)

27. A two-index tensor $T^{\mu\nu}$ or $T_{\mu\nu}$ is *diagonal* if the only non-zero components have $\mu = \nu$.

(a) Show that if the metric $g_{\mu\nu}$ is diagonal, then for any diagonal tensor $T^{\mu\nu}$ we will have $T^{\mu}{}_{\nu} = T^{\hat{\mu}}{}_{\hat{\nu}}$, *i.e.*, when written with one index down and one up, it will be the same thing. What is $T^{\mu}{}_{\nu}$ for a perfect fluid at rest?

(b) For a general spherically symmetric time-independent metric,

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

use conservation of the stress-energy tensor, $\nabla_{\mu} T^{\mu}{}_{\nu} = 0$ with $\nu = r$ to show that

$$\partial_r P = -\tfrac{1}{2}(\rho + P)\partial_r(\ln f)$$

28. In standard coordinates, the Schwarzschild metric takes the form

$$ds^2 = -(1 - 2GM/r)dt^2 + (1 - 2GM/r)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Change coordinates to a new radial coordinate *R* defined by

$$r = R + GM + \frac{G^2M^2}{4R} = \left(R + \tfrac{1}{2}GM\right)^2 \Big/ R$$

Show that in this new coordinate system, the metric can be rewritten as

$$ds^2 = -A(R)dt^2 + B(R)\left[dR^2 + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

and determine the new metric functions $A(R)$ and $B(R)$.