Problems 11-15

11. A pion at rest (mass = m_π) decays to a muon (mass = m_μ) and a neutrino (mass = 0). Work out the energy of the muon after the decay.

12. Work out explicitly how the components of the electric and magnetic field mix under (a) a rotation and (b) a Lorentz boost, as given by the following formulas:

$$\Lambda^\alpha{}_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \Lambda^\alpha{}_\beta = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

13. Show that $\nabla_\alpha F^{\beta\alpha} = j^\beta / \varepsilon_0$ in Cartesian coordinates automatically implies the conservation of charge $\nabla_\beta j^\beta = 0$.

14. The stress-energy tensor $T^{\alpha\beta}$ is a symmetric tensor, $T^{\alpha\beta} = T^{\beta\alpha}$.
    (a) Work out how the components of the rotated tensor $\bar{T}^{\alpha\beta}$ are related to those in the original frame, if we perform a rotation by 90 degrees around the *z*-axis (the first Lorentz transformation in problem 12). You may find it easiest to leave your answer in matrix notation.
    (b) Show that if the stress-energy tensor is unchanged by the rotation in part (a), we must have $T^{tx} = T^{ty} = T^{xy} = T^{xz} = T^{yz} = 0$ and $T^{xx} = T^{yy}$.

15. Scalar quantities should be agreed on by all observers. Work out the scalar quantities $F_{\alpha\beta}F^{\alpha\beta}$ and $\varepsilon_{\alpha\beta\delta\gamma}F^{\alpha\beta}F^{\delta\gamma}$ in terms of the electric and magnetic fields. Then use the first of these to show that the electromagnetic energy density, $\rho = \frac{1}{2}\varepsilon_0\left(\mathbf{E}^2 + \mathbf{B}^2\right)$ can be written in the form

$$T^{tt} = \varepsilon_0\left(F^t{}_\mu F^{t\mu} - \tfrac{1}{4}\eta^{tt}F^{\nu\mu}F_{\nu\mu}\right).$$