Problems 44-47

44. In class, when demonstrating conservation of the stress-energy tensor, I used the relationship

$$\tfrac{1}{2}T^{\mu\nu}\left(\xi^{\alpha}\partial_{\alpha}g_{\mu\nu}+g_{\alpha\nu}\partial_{\mu}\xi^{\alpha}+g_{\mu\alpha}\partial_{\nu}\xi^{\alpha}\right)=T^{\mu\nu}\nabla_{\mu}\xi_{\nu}$$

Fill in the missing steps.

45. Show that the variation of the inverse metric is related to the variation of the metric by $\delta g^{\mu\nu}=-g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}$.

46. The Lagrangian density for the electromagnetic field is $\mathcal{L}=-\tfrac{1}{4}\varepsilon_0 F_{\mu\nu}F^{\mu\nu}$, where the fields *F* are defined as $F_{\mu\nu}\equiv\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}=\nabla_{\mu}A_{\nu}-\nabla_{\nu}A_{\mu}$ and $F^{\mu\nu}\equiv g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}$. Show that the Euler-Lagrange equations yield the standard source-free Maxwell equations, $\varepsilon_0\nabla_{\mu}F^{\mu\nu}=0$.

47. Show that for the Lagrangian density in problem 46, the stress-energy tensor for the electromagnetic field is $T^{\mu\nu}=\varepsilon_0\left(F^{\mu}{}_{\alpha}F^{\nu\alpha}-\tfrac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)$