Problems 1-5

1. Define a dot product for 1-forms $\tilde{\omega}^i \cdot \tilde{\omega}^j \equiv g^{ij}$

   (a) Show that $\tilde{u} \cdot \tilde{v} \equiv \langle \tilde{u}, \vec{v} \rangle$ implies $u^i = g^{ij} u_j$

   (b) Show that if we think of g_{ij} as a matrix, then the matrix $g^{ij} = \left(g_{ij}\right)^{-1}$.

   hint: $M^{-1} \cdot M = \mathbf{1}$

2. Consider a 2d skew Cartesian coordinates, related to conventional coordinates by

$$x = \bar{x} + \bar{y} \sin\theta$$
$$y = \bar{y} \cos\theta$$

   (a) Express $d\bar{x}^i$ in terms of dx^i.
   (b) Express $\partial_{\bar{x}^i}$ in terms of ∂_{x^i}
   (c) Find the components of g_{ij} and g^{ij}.
   (d) Find the lengths of $\partial_{\bar{x}}$ and $\partial_{\bar{y}}$ and the angle between them.
   (e) Draw the vectors ∂_x, ∂_y, $\partial_{\bar{x}}$ and $\partial_{\bar{y}}$ on a diagram of the coordinate system at one point.

3. Spherical coordinates are defined by

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

   (a) Show that $g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$

   (b) What is the volume element d^3V in spherical coordinates?

4. Show that $\nabla_i \alpha_j = \partial_i \alpha_j - \Gamma^k_{ji} \alpha_k$. Use Leibnitz's rule, $\nabla_i \left(AB\right) = \left(\nabla_i A\right) B + A \nabla_i B$.

5. Use the fact that $\nabla_i v^j$ transforms as a tensor to prove one of the following two relationships: