Physics 780 – General Relativity

Homework Set W

54. [10] In class I assumed the identity $\int T^{0i}(x)x^j d^3\mathbf{x} = \int T^{0j}(x)x^i d^3\mathbf{x}$ **. This isn't quite true, but it is close to true.**

(a) Call the difference between these two expressions J^{ij}**. Write an expression for the time derivative of** J^{ij}**.**

(b) Use the identity $\partial_0 T^{0\alpha} = -\partial_k T^{k\alpha}$ **to rewrite the integrals as space derivatives.**

This is pretty straightforward. We have

$$\partial_0 J^{ij} = \int \left[T^{0i}(x)x^j - T^{0j}(x)x^i \right] d^3\mathbf{x} = \int \left[\partial_0 T^{0i}(x)x^j - \partial_0 T^{0j}(x)x^i \right] d^3\mathbf{x}$$
$$= \int \left[-\partial_k T^{ki}(x)x^j - \partial_k T^{kj}(x)x^i \right] d^3\mathbf{x}.$$

(c) Integrate these expression by parts. Since we are going to assume ***T*** **vanishes at sufficient distances, the surface terms vanish. Simplify the remaining derivatives by using** $\partial_k x^\ell = \delta_k^\ell$**, and do the sum over** ***k*****.**

Integrating by parts, and ignoring surface terms, we have

$$\partial_0 J^{ij} = \int_s \left[-T^{ki}(x)x^j + T^{kj}(x)x^i \right] dS + \int \left[T^{ki}(x)\partial_k x^j - T^{kj}(x)\partial_k x^i \right] d^3\mathbf{x}$$
$$= \int \left[T^{ki}(x)\delta_k^j - T^{kj}(x)\delta_k^i \right] d^3\mathbf{x} = \int \left[T^{ji}(x) - T^{ij}(x) \right] d^3\mathbf{x} = 0.$$

We note that at the last step we had to take advantage of the fact that T^{ij} is symmetric. This is one of the reasons it is so desirable to have a symmetric stress-energy tensor.

(d) Show that J^{ij} **is constant. Since we are focusing on the portion of the integrals that oscillate, this means that any oscillating component satisfies** $\int T^{0i}(x)x^j d^3\mathbf{x} = \int T^{0j}(x)x^i d^3\mathbf{x}$ **.**

We basically just proved it is constant, which means it has no oscillating part.

55. [15] We had some angular integrals that needed to be done, of the form $\int d\Omega$, $\int k_i k_j d\Omega$ and $\int k_i k_j k_\ell k_m d\Omega$, where $\int d\Omega \equiv \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta$.

(a) Find $\int d\Omega$. This part of the problem is completely different from the remaining parts.

We simply do the easy integral, which is

$$\int d\Omega = \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi = -\cos\theta\Big|_0^{\pi} \left(2\pi\right) = 2 \cdot \pi = 4\pi .$$

(b) To find $\int k_i k_j d\Omega$, first note that since all directions are created equal, it must be some sort of invariant tensor. The only tensors in 2D that are invariant are δ_{ij} and $\tilde{\varepsilon}_{ijk}$ and combinations of them. Argue that the result must be proportional to δ_{ij}. Call the constant of proportionality *A*.

This integral must be an invariant tensor with two indices; the only such tensor is δ_{ij}. It also must be symmetric, which it is. Hence $\int k_i k_j d\Omega = A\delta_{ij}$.

(c) Multiply the integral in part (b) by δ_{ij} summing over *i* and *j*, and use the identity $\mathbf{k}^2 = \omega^2$ to simplify. Determine *A*.

We have

$$\begin{aligned}
&\delta_{ij} \int k_i k_j d\Omega = A\delta_{ij}\delta_{ij} , \\
&\int \mathbf{k}^2 d\Omega = 3A, \\
&4\pi\omega^2 = 3A, \\
&A = \tfrac{4}{3}\pi\omega^2 .
\end{aligned}$$

(d) To find $\int k_i k_j k_\ell k_m d\Omega$, first note that since all directions are created equal, it must be some sort of invariant tensor. Argue that the result must be proportional to $\delta_{ij}\delta_{\ell m} + \delta_{i\ell}\delta_{jm} + \delta_{im}\delta_{j\ell}$. Call the constant of proportionality *B*.

This time we need an invariant tensor with four indices. The only possible tensors are $\delta_{ij}\delta_{\ell m}$, $\delta_{i\ell}\delta_{jm}$ and $\delta_{im}\delta_{j\ell}$. However, it must also be symmetric under any interchange of indices, and the only such tensor is $\delta_{ij}\delta_{\ell m} + \delta_{i\ell}\delta_{jm} + \delta_{im}\delta_{j\ell}$, so

$$\int k_i k_j k_\ell k_m d\Omega = B\left(\delta_{ij}\delta_{\ell m} + \delta_{i\ell}\delta_{jm} + \delta_{im}\delta_{j\ell}\right).$$

(e) Do something similar to what you did in part (c) and use the identity $\mathbf{k}^2 = \omega^2$ to simplify and determine *B*.

This time it is not clear what to do, but we can try multiplying by δ_{ij} and see if it works:

$$\delta_{ij}\int k_i k_j k_\ell k_m d\Omega = B\left(\delta_{ij}\delta_{\ell m}\delta_{ij} + \delta_{i\ell}\delta_{jm}\delta_{ij} + \delta_{im}\delta_{j\ell}\delta_{ij}\right),$$

$$\int \mathbf{k}^2 k_\ell k_m d\Omega = B\left(3\delta_{\ell m} + \delta_{\ell m} + \delta_{\ell m}\right),$$

$$\omega^2 \int k_\ell k_m d\Omega = 5B\delta_{\ell m},$$

$$\omega^2 \tfrac{4}{3}\pi\omega^2\delta_{\ell m} = 5B\delta_{\ell m},$$

$$B = \tfrac{4}{15}\pi\omega^4.$$

We used the integral found in part (b) at the penultimate step. We therefore have

$$\int k_i k_j k_\ell k_m d\Omega = \tfrac{4}{15}\pi\omega^4\left(\delta_{ij}\delta_{\ell m} + \delta_{i\ell}\delta_{jm} + \delta_{im}\delta_{j\ell}\right).$$