Physics 780 – General Relativity

Homework Set M

Assume you are working in the Schwarzschild metric for all of these problems.

31. Consider a massive particle moving in this metric.
    (a) Show that if at any time $U^\theta = U^\phi = 0$, then from the geodesic equations we will also have $dU^\theta/d\tau = dU^\phi/d\tau = 0$, so this will continue to be true indefinitely.
    (a) Show that if $\theta = \frac{1}{2}\pi$ and $U^\theta = 0$, then from the geodesic equations we will also have $dU^\theta/d\tau = 0$, so this will continue to be true indefinitely.

32. Consider a massive particle (so $U_\mu U^\mu = -1$) starting at rest near $r = \infty$, so $U^\theta = U^\phi = 0$.
    (a) If $U^r = 0$ at infinity, what is the value of U^t and of $U_t = -E$? Recall that E is a constant.
    (b) Find a formula for U^t and for U^r as a function of r.
    (c) Take your formula from part (b) for $U^r = dr/d\tau$ and integrate it over radius to find out how much proper time it takes to fall from a distance r down to $r = 0$. Assume the formulas work right through $r = 2GM$, despite the apparent singularity of the metric there.
    (d) How long does it take after you cross the Schwarzschild radius $r = 2GM$ to reach the origin for the black hole at the center of our galaxy, with a mass of 4.4×10^6 solar masses. The Schwarzschild radius for the Sun was found in problem set A, and is 2.95 km.

33. Consider a massive particle of arbitrary energy moving in the $\theta = \frac{1}{2}\pi$ plane.
    (a) A circular orbit is possible whenever there is a local maximum or local minimum of the effective potential. Find a formula for the two radii where circular orbits are possible in terms of J and M. Which of these is stable, and which unstable?
    (b) As you increase J, the two radii found in part (a) move together and merge. What is the value of J, and the corresponding value of r when this happens, in terms of M? This is called the innermost stable circular orbit, or ISCO for this metric.