Physics 780 – General Relativity

# Homework Set F

14. The metric in flat 3D space is $ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. Show that in spherical coordinates, this is given by $ds^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$. Spherical coordinates are defined by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

15. Consider the 3D metric $ds^2 = \left(1-r^2\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2$. This has an *apparent* singularity at $r = 0$, because the determinant $g = 0$ there, but we understand that that's just a coordinate singularity at a point. What about the apparent singularity at $r = 1$, where $g = \infty$?
    (a) By looking, for example, at the circle defined by $r \to 1$, $\theta = \tfrac{1}{2}\pi$, $\phi = \left(0, 2\pi\right)$ argue that those points with $r = 1$ are decidedly not just a point.
    (b) Make the substitution $r = \sin\psi$, while keeping the coordinates $\theta$ and $\phi$. Show that the resulting metric no longer has a singularity at $r = 1$ (now at $\psi = \tfrac{1}{2}\pi$).
    (c) There is now an apparent singularity where the metric has problems at $\psi = \pi$. Convince yourself that this is, in fact, just a point.

16. Given a metric, is it curved? The answer isn't always obvious. Consider the metric

$$ds^2 = -\mathrm{d}t^2 + t^2\left(\frac{\mathrm{d}r^2}{1+r^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right)$$

We will convert to coordinates $\left(t, r, \theta, \phi\right) \to \left(T, R, \theta, \phi\right)$, defined by

$$\left\{\begin{matrix} T = t\sqrt{r^2+1} \\ R = rt \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} t = \sqrt{T^2 - R^2} \\ r = \dfrac{R}{\sqrt{T^2-R^2}} \end{matrix}\right\}$$

with $\theta$ and $\phi$ the same in both coordinate systems
    (a) Write $\mathrm{d}t$ and $\mathrm{d}r$ in terms of $\mathrm{d}T$ and $\mathrm{d}R$.
    (b) Write the metric out entirely in the new coordinates. If you make no mistake, there should be no cross-terms and the coefficients should all be simple.
    (c) By comparison with the metric in problem 14, argue this is simply disguised flat spacetime.