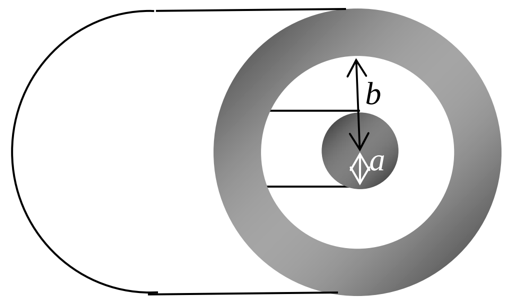
Physics 712 – Electricity and Magnetism

Solutions to Final Exam, Spring 2016

Please note that some possibly helpful formulas appear on the second page. The number of points on each problem and part is marked in square brackets at the beginning of the problem.



1. [30] A capacitor consists of two coaxial cylindrical conductors, the inner one with outer radius *a*, and the outer with inner radius *b*, and having length *L* much larger than *a* or *b*.

(a) [10] Find the capacitance of the capacitor.

Capacitance is defined by the equation $Q = C\Delta V$. If we have a charge Q on the inner conductor, then the electric field (away from the ends) will be radial outward $\mathbf{E} = E(\rho)\hat{\boldsymbol{\rho}}$, and we then we can use Gauss's law with a cylinder of radius ρ and length L,

$$Q/\varepsilon_0 = 2\pi\rho LE,$$

$$E = \frac{Q}{2\pi\varepsilon_0\rho L}$$

We can then integrate this to get the potential

$$\Delta V = \int_a^b E(\rho)\,d\rho = \frac{Q}{2\pi\varepsilon_0 L}\int_a^b \frac{d\rho}{\rho} = \frac{Q}{2\pi\varepsilon_0 L}\ln(b/a),$$

$$C = \frac{Q}{\Delta V} = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}.$$

(b) [7] A battery with voltage difference ΔV is now attached to the capacitor. Find the amount of charge the battery sends to the capacitor, and the amount of energy required to charge the capacitor.

Of course, $Q = C\Delta V$, and the energy can be found from $U = \tfrac{1}{2}Q\Delta V$, so we have

$$Q = C\Delta V = \frac{2\pi\varepsilon_0 L\Delta V}{\ln(b/a)},$$

$$U = \tfrac{1}{2}Q\Delta V = \frac{\pi\varepsilon_0 L(\Delta V)^2}{\ln(b/a)}.$$

(c) [13] While still connected to the battery, a dielectric material of permittivity $\varepsilon = 2\varepsilon_0$ is slid between the two conductors. How much work does the battery do maintaining the potential, how much does the capacitor energy increase, and how much work must be done to push the dielectric in?

Obviously, doubling the permittivity doubles the capacitance, which since the voltage stays the same, doubles the charge. Therefore, the charge changes by an amount $\Delta Q = Q$, and since it is moving across a constant voltage difference, the work done is

$$W_B = (\Delta Q)(\Delta V) = Q(\Delta V) = \frac{2\pi\varepsilon_0 L(\Delta V)^2}{\ln(b/a)}.$$

The capacitor stayed at constant voltage, so the new capacitor energy is $U' = \frac{1}{2}(2Q)\Delta V = Q\Delta V$, so the change in energy is

$$\Delta U = U' - U = 2U - U = U = \frac{\pi\varepsilon_0 L(\Delta V)^2}{\ln(b/a)}.$$

This is only half as large as the energy put in. Where did the balance in energy go? It went into drawing the dielectric into the capacitor. The work done by us on the dielectric is

$$W_D = \Delta U - W_B = \frac{\pi\varepsilon_0 L(\Delta V)^2}{\ln(b/a)} - \frac{2\pi\varepsilon_0 L(\Delta V)^2}{\ln(b/a)} = -\frac{\pi\varepsilon_0 L(\Delta V)^2}{\ln(b/a)}.$$

It is negative, indicating that, in fact, the dielectric is being drawn into the battery.

2. [15] In the presence of a charged plasma, it is possible to create electromagnetic waves that are longitudinal, having electric polarization parallel to the direction of propagation. The fields take the form $\mathbf{E}(\mathbf{x},t) = E_0\hat{\mathbf{x}}\cos(kx-\omega t)$, but with no magnetic field $\mathbf{B} = 0$.

(a) [6] Show that this electric field (and lack of magnetic field) can be written purely in terms of a vector potential $\mathbf{A}(\mathbf{x},t)$, without the use of the scalar potential, so that $\Phi = 0$.

Since $\Phi = 0$, we have $\mathbf{E} = -\dot{\mathbf{A}}$, so we can get **A** by performing a time integral of **E**:

$$\mathbf{A}(\mathbf{x},t) = -\int \mathbf{E}(\mathbf{x},t)\,dt = -E_0\hat{\mathbf{x}}\int \cos(kx-\omega t)\,dt = \frac{E_0}{\omega}\hat{\mathbf{x}}\sin(kx-\omega t).$$

The constant of integration is irrelevant, since we are simply trying to find a vector potential **A** that works.

(b) [9] Convince yourself that this is *not* Coulomb gauge. Find a gauge transformation that causes A to vanish. Find the scalar potential Φ' in this gauge.

For this vector potential, $\nabla\cdot\mathbf{A} = E_0 k\omega^{-1}\cos(kx-\omega t)$, but Coulomb gauge is defined by $\nabla\cdot\mathbf{A} = 0$. Clearly, not Coulomb gauge. To make **A** vanish, we need to find a gauge function Λ such that $\mathbf{A}' = \mathbf{A} + \nabla\Lambda = 0$, so we need

$$\nabla\Lambda = -\mathbf{A} = -\frac{E_0}{\omega}\hat{\mathbf{x}}\sin(kx-\omega t).$$

Since we want the result to be in the *x*-direction, we clearly want a function of *x* that when differentiated yields sine. It's pretty obvious that cosine will work. In fact, if we pick

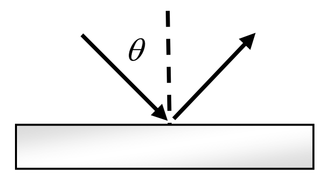
$$\Lambda(\mathbf{x},t) = \frac{E_0}{\omega k}\cos(kx-\omega t),$$

then when we perform a gauge transformation, we will have

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda = \frac{E_0}{\omega}\hat{\mathbf{x}}\sin(kx-\omega t) + \nabla\frac{E_0}{\omega k}\cos(kx-\omega t) = 0,$$

$$\Phi' = \Phi + \frac{\partial}{\partial t}\frac{E_0}{\omega k}\cos(kx-\omega t) = \frac{E_0}{k}\sin(kx-\omega t).$$

3. [20] An electromagnetic wave with wave number magnitude *k* is incident from vacuum on a perfect conductor at an angle θ with respect to the normal, as sketched at right. Let the vertical direction be *z* and the horizontal direction *x*. Let the incident wave have electric field amplitude $\mathbf{E}_0$.



(a) [6] Write, in complex notation, the general form for the incident electric and magnetic fields as functions of space and time. You must write any constants you use, like ω, in terms of given constants.

In general, a wave takes the form

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}, \quad \mathbf{B} = \frac{1}{\omega}\mathbf{k}\times\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}.$$

The vector $\mathbf{E}_0$ is restricted to be perpendicular to $\mathbf{k}$. The direction of $\mathbf{k}$ is specified by the problem, and in vacuum, $\omega = ck$, so

$$\mathbf{k} = k\sin\theta\hat{\mathbf{x}} - k\cos\theta\hat{\mathbf{z}} \quad \text{and} \quad \omega = ck.$$

(b) [8] Write the general form of the reflected wave $\mathbf{E}'$ and $\mathbf{B}'$ as functions of space and time in terms of the reflected electric field amplitude $\mathbf{E}'_0$. Explain logically why you were forced to pick the values you pick for the reflected wave number vector $\mathbf{k}'$ (i.e., why is the law of reflection valid?).

The reflected wave will take the same general form, so

$$\mathbf{E}' = \mathbf{E}'_0 e^{i\mathbf{k}'\cdot\mathbf{x}-i\omega t}, \quad \mathbf{B} = \frac{1}{\omega}\mathbf{k}'\times\mathbf{E}'_0 e^{i\mathbf{k}'\cdot\mathbf{x}-i\omega t}.$$

The reflected wave will have to satisfy appropriate boundary conditions at $z = 0$, so to make it match with the incoming weave, we are going to have to pick the same frequency. Also, to make it match at all x, we are going to have to make $k_x = k'_x$. Finally, we are going to still have $\omega = ck'$, which implies $k = k'$, so that $k_x^2 + k_z^2 = k_x'^2 + k_z'^2$, or since $k_x = k'_x$, we will have $k_z^2 = k_z'^2$, so that $k'_z = \pm k_z$. But we want the wave to be outgoing, so we see that

$$\mathbf{k}' = k\sin\theta\hat{\mathbf{x}} + k\cos\theta\hat{\mathbf{z}}$$

This implies the law of reflection, that the reflected wave has the same angle compared to the normal as the incident wave.

(c) [6] Assuming that the incident wave has $\mathbf{E}_0 = E_0\hat{\mathbf{y}}$, find $\mathbf{E}'_0$.

The parallel electric field must vanish on the boundary $z = 0$, which implies

$$0 = \mathbf{E}_0 e^{ik_x x-i\omega t} + \mathbf{E}'_0 e^{ik_x x-i\omega t}$$

at least parallel to the surface of the conductor. We therefore have $\mathbf{E}'_0 = -\mathbf{E}_0 = -E_0\hat{\mathbf{y}}$, since this is parallel to the conductor.

4. [10] How do the quantities $\mathbf{E}^2 - c^2\mathbf{B}^2$ and $\mathbf{E}\cdot\mathbf{B}$ transform under (a) proper rotations, (b) improper rotations, and (c) time reversal?

Under proper rotations, the dot product of two vectors is a scalar, so both of these are scalars.

Under improper rotations, say reflections, **E** does exactly what you'd think a vector would do, which leaves $\mathbf{E}^2$ unchanged, but **B** picks up an extra minus sign, which means that $\mathbf{B}^2$ is also unchanged. So $\mathbf{E}^2 - c^2\mathbf{B}^2$ is still a scalar under improper rotations. However, $\mathbf{E}\cdot\mathbf{B}$ will have **E** reflect normally, but **B** picks up a minus sign, so $\mathbf{E}\cdot\mathbf{B}$ will change sign. Hence $\mathbf{E}\cdot\mathbf{B}$ is a pseudoscalar.

Under time reversal, **E** stays the same and **B** changes sign. Hence $\mathbf{E}^2 - c^2\mathbf{B}^2$ will be unchanged under time reversal, while $\mathbf{E}\cdot\mathbf{B}$ stays the same.

5. [10] How does the quantity $\mathbf{E}\cdot\mathbf{B}$ transform under a Lorentz boost in the *x*-direction?

When we perform a Lorentz boost, the fields transform according to

$$E'_x = E_x\,,\quad B'_x = B_x$$

$$\mathbf{E}'_\perp = \gamma\left(\mathbf{E}_\perp + v\hat{\mathbf{x}}\times\mathbf{B}\right) = \gamma\left[E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}} + v\hat{\mathbf{x}}\times\left(B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}\right)\right] = \gamma\left[\left(E_y - vB_z\right)\hat{\mathbf{y}} + \left(E_z + vB_y\right)\hat{\mathbf{z}}\right]$$

$$\mathbf{B}'_\perp = \gamma\left(\mathbf{B}_\perp - \mathbf{v}\times\mathbf{E}/c^2\right) = \gamma\left[B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}} - \frac{v}{c^2}\hat{\mathbf{x}}\times\left(E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}\right)\right] = \gamma\left[\left(B_y + \frac{v}{c^2}E_z\right)\hat{\mathbf{y}} + \left(B_z - \frac{v}{c^2}E_y\right)\hat{\mathbf{z}}\right]$$

We therefore have

$$\begin{aligned}
\mathbf{E}'\cdot\mathbf{B}' &= E'_xB'_x + E'_yB'_y + E'_zB'_z = E_xB_x + \gamma^2\left(E_y - vB_z\right)\left(B_y + \frac{v}{c^2}E_z\right) + \gamma^2\left(E_z + vB_y\right)\left(B_z - \frac{v}{c^2}E_y\right) \\
&= E_xB_x + \gamma^2\left(E_yB_y - vB_zB_y + \frac{v}{c^2}E_yE_z - \frac{v^2}{c^2}B_zE_z + E_zB_z + vB_yB_z - \frac{v}{c^2}E_zE_y - \frac{v^2}{c^2}B_yE_y\right) \\
&= E_xB_x + \gamma^2\left(E_yB_y - \frac{v^2}{c^2}B_yE_y + E_zB_z - \frac{v^2}{c^2}B_zE_z\right) = E_xB_x + \gamma^2\left(1 - \frac{v^2}{c^2}\right)\left(E_yB_y + E_zB_z\right) \\
&= E_xB_x + E_yB_y + E_zB_z = \mathbf{E}\cdot\mathbf{B}\,.
\end{aligned}$$

So, in fact, this expression is unchanged.

6. [30] A particle with charge *q* and mass *m*, initially at rest at $\tau = 0$ is placed in a uniform electric field $\mathbf{E} = E\hat{\mathbf{x}}$.

(a) [8] Find a set of coupled differential equations for the derivatives of the four-velocity U^μ as functions of time.

We start by writing the electric field, first with indices both up, and then lower the second index by changing the three columns corresponding to the space coordinates:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E/c & 0 & 0 \\ E/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F^\mu{}_\nu = \begin{pmatrix} 0 & E/c & 0 & 0 \\ E/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We then use the equation $m\left(dU^\mu/d\tau\right) = qF^\mu{}_\nu U^\nu$. We therefore have

$$m\frac{d}{d\tau}\begin{pmatrix} U^0 \\ U^1 \\ U^2 \\ U^3 \end{pmatrix} = \frac{q}{c}\begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} U^0 \\ U^1 \\ U^2 \\ U^3 \end{pmatrix}.$$

We then write these equations out as

$$\frac{dU^0}{d\tau} = \frac{qE}{mc}U^1, \quad \frac{dU^1}{d\tau} = \frac{qE}{mc}U^0, \quad \frac{dU^2}{d\tau} = \frac{dU^3}{d\tau} = 0.$$

(b) [12] Find the solution of the equations from part (a) subject to appropriate boundary conditions at $\tau = 0$.

At $\tau = 0$, the particle has $U^\mu = \left(c, 0, 0, 0\right)$. The last two equations are trivial to solve, $U^2\left(\tau\right) = U^3\left(\tau\right) = 0$. To solve the other equations, take another derivative of the first equation, which then yields

$$\frac{d^2U^0}{d\tau^2} = \frac{qE}{mc}\frac{d}{d\tau}U^1 = \frac{q^2E^2}{m^2c^2}U^0.$$

This has general solution

$$U^0\left(\tau\right) = A\exp\left(\frac{qE\tau}{mc}\right) + B\exp\left(-\frac{qE\tau}{mc}\right).$$

We can substitute these into, say, the first of the differential equations, we find

$$U^1\left(\tau\right) = \frac{mc}{qE}\frac{d}{d\tau}U^0\left(\tau\right) = A\exp\left(\frac{qE\tau}{mc}\right) - B\exp\left(-\frac{qE\tau}{mc}\right).$$

Since $U^1(0) = 0$, we conclude that $A = B$. Since $U^0(0) = c$, we have $A + B = c$, which then tells you $A = B = \frac{1}{2}c$. We therefore have

$$U^0(\tau) = \frac{c}{2}\exp\left(\frac{qE\tau}{mc}\right) + \frac{c}{2}\exp\left(-\frac{qE\tau}{mc}\right) = c\cosh\left(\frac{qE\tau}{mc}\right), \quad U^1(\tau) = c\sinh\left(\frac{qE\tau}{mc}\right).$$

(c) [10] A proton has mass $m = 1.673\times10^{-27}$ kg $= 938.3$ MeV/c^2 and charge $q = e = 1.602\times10^{-19}$ C. It is placed in a constant electric field $E = 1.000$ V/m. How much proper time τ (in seconds) will it take to reach 99% of the speed of light?

The four-velocity is given by $U^\mu(\tau) = \gamma(c, \mathbf{v})$, so that $\mathbf{v}/c = \mathbf{U}/U^0$, so in this case, we have

$$\frac{v}{c} = \frac{U^1(\tau)}{U^0(\tau)} = \frac{\sinh(qE\tau/mc)}{\cosh(qE\tau/mc)} = \tanh\left(\frac{qE\tau}{mc}\right).$$

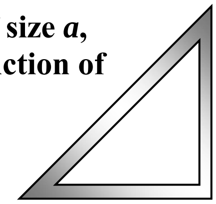
We therefore have

$$\frac{qE\tau}{mc} = \tanh^{-1}\left(\frac{v}{c}\right) = \tanh^{-1}(0.99) = 2.647,$$

$$\tau = \frac{2.647mc}{qE} = \frac{2.647\left(938.3\times10^6 \text{ eV}/c^2\right)c}{e(1 \text{ V/m})} = \frac{2.647\left(9.383\times10^8 \text{ m}\right)}{c}$$

$$= \frac{2.647\left(9.383\times10^8 \text{ m}\right)}{2.998\times10^8 \text{ m/s}} = 8.28 \text{ s}.$$

7. **[35] A waveguide is in the shape of 45-45-90 triangle, empty, with legs of size *a*, with boundaries at *y* = 0, *x* = *a*, and *y* = *x*. Consider a potential mode function of the form**



$$\psi(x,y) = B\cos(\alpha x) + C\cos(\alpha y).$$

(a) [15] By considering the boundary at *y* = 0, determine if this would work for a TE or TM mode. By considering the other two boundaries, find suitable conditions on α, *B*, and *C*.

For TM modes, $\psi = E_z$, and since $\mathbf{E}_\parallel = 0$ on the surface of a conductor, and E_z is parallel, but clearly the mode function does not vanish at $y = 0$. We therefore expect we must have a TE mode, for which we demand $\partial\psi/\partial n = 0$. In fact, at $y = 0$, we have

$$\frac{\partial\psi}{\partial n} = \frac{\partial\psi}{\partial y} = -\alpha C\sin(\alpha y)\Big|_{y=0} = 0.$$

So we are dealing with a TE mode.

We also must have the boundaries at $x = a$ and $x = y$ work. Looking at $x = a$, we have

$$0 = \left.\frac{\partial\psi}{\partial x}\right|_{x=a} = -B\alpha\sin(\alpha x)\Big|_a = -B\alpha\sin(\alpha a)$$

This suggests that $a\alpha$ must be an integer multiple of pi, so

$$\alpha = \frac{\pi n}{a}.$$

This only works when *n* is a positive integer. Finally, we have to get appropriate boundary conditions on the boundary $x = y$. The perpendicular to this line is in the direction $\hat{\mathbf{x}} - \hat{\mathbf{y}}$, so we have

$$0 = \frac{\partial\psi}{\partial n} = \frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial y} = \left[-\alpha B\sin(\alpha x) + \alpha C\sin(\alpha x)\right]\Big|_{x=y} = \alpha(C-B)\sin(\alpha x).$$

To make this true all along the boundary, we must pick $B = C$.

(b) [7] Check that it satisfies the mode equation, and determine γ.

The mode equation is $\left(\nabla_t^2 + \gamma^2\right)\psi = 0$. So we have

$$0 = \left(\nabla_t^2 + \gamma^2\right)\psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right)\psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right)B\left[\cos(\alpha x) + \cos(\alpha y)\right]$$
$$= B\left[-\alpha^2\cos(\alpha x) - \alpha^2\cos(\alpha y) + \gamma^2\cos(\alpha x) + \gamma^2\cos(\alpha y)\right] = \left(\gamma^2 - \alpha^2\right)\psi.$$

It is obvious from this formula that $\gamma = \alpha = \dfrac{\pi n}{a}$.

(c) [8] Write out explicitly the wave function E_z or B_z (which ever is appropriate) as a function of space and time, if it has wave number k in the z-direction, explicitly giving an expression for the frequency ω.

The space and time dependence for the magnetic field (since this is a TE mode) will be

$$B_z = \psi(x, y) e^{ikz - i\omega t} = B\left[\cos\left(\frac{\pi n x}{a}\right) + \cos\left(\frac{\pi n y}{a}\right)\right] e^{ikz - i\omega t} .$$

The frequency ω is given by

$$\varepsilon_0 \mu_0 \omega^2 = k^2 + \gamma^2 = k^2 + \frac{\pi^2 n^2}{a^2}$$

$$\omega = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \sqrt{k^2 + \frac{\pi^2 n^2}{a^2}} = c\sqrt{k^2 + \frac{\pi^2 n^2}{a^2}} .$$

(d) [5] We now modify this waveguide by making it into a cavity by adding conductors at $z = 0$ and $z = a$. What restrictions does this impose? Find a formula for the frequencies ω, and find the lowest frequencies.

The magnetic field B_z now takes the form $B_z = \psi(x, y)\sin(kz)e^{-i\omega t}$, where we chose sine because we need $\dot{B}_z = 0$ on this boundary. We also need it to vanish at $z = a$, so again, $a\alpha$ must be a multiple of pi again. We therefore have

$$k = \frac{\pi p}{a},$$

where again we have p as a positive integer. The frequency for these states is therefore

$$\omega = c\sqrt{\alpha^2 + k^2} = c\sqrt{\frac{\pi^2 n^2}{a^2} + \frac{\pi^2 p^2}{a^2}} = \frac{\pi c}{a}\sqrt{n^2 + p^2} .$$

The lowest state has frequency $\pi c\sqrt{2}/a$.