Physics 712 – Electricity and Magnetism

Solutions to Midterm Exam, Spring 2016

Please note that some possibly helpful formulas and integrals appear on the second page. Each question is worth 20 points.

1. A charge distribution in vacuum produces a potential given, in cylindrical coordinates, by $\Phi(\rho,\phi,z)=\frac{\alpha}{\varepsilon_0}\ln\left(1+\frac{a^2}{\rho^2}\right)$; that is, it is independent of both *z* and ϕ. Find the electric field E everywhere and the charge density $\rho(\mathbf{x})$ everywhere away from the *z*-axis. Don't confuse the charge density with the cylindrical coordinate ρ. Demonstrate that there is also a linear charge density λ along the *z*-axis, and determine its magnitude.

We find the electric field from the equation

$$\mathbf{E}=-\nabla\Phi=-\hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho}\Phi-\hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial}{\partial\phi}\Phi-\hat{\mathbf{z}}\frac{\partial}{\partial z}\Phi=-\frac{\alpha}{\varepsilon_0}\hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho}\ln\left(1+\frac{a^2}{\rho^2}\right)=-\frac{\alpha\hat{\boldsymbol{\rho}}}{\varepsilon_0\left(1+a^2/\rho^2\right)}\frac{-2a^2}{\rho^3}$$

$$=\frac{2a^2\alpha\hat{\boldsymbol{\rho}}}{\varepsilon_0\rho\left(\rho^2+a^2\right)}.$$

We then find the charge density using

$$\rho(\mathbf{x})=\nabla\cdot\mathbf{D}=\varepsilon_0\nabla\cdot\mathbf{E}=\varepsilon_0\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho E_\rho\right)+\frac{1}{\rho}\frac{\partial}{\partial\phi}E_\phi+\frac{\partial}{\partial z}E_z\right]=\frac{2a^2\alpha}{\rho}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho^2+a^2}\right)=\frac{-4a^2\alpha}{\left(\rho^2+a^2\right)^2}.$$

However, these formulas are a little tricky to apply at $\rho=0$, which corresponds to the *z*-axis. To find the charge along the *z*-axis, draw a small cylinder of radius ρ and length *L* centered on the *z*-axis. The total charge inside this cylinder will be $q(V)=\varepsilon_0\int_S\mathbf{E}\cdot\hat{\mathbf{n}}\,da$. Because the electric field is in the radial direction, only the lateral surface will contribute. This surface has an area of $2\pi\rho L$, so we have

$$q(V)=\varepsilon_0\int_S\mathbf{E}\cdot\hat{\mathbf{n}}\,da=2\pi\rho L\frac{2a^2\alpha}{\rho\left(\rho^2+a^2\right)}=\frac{4\pi\alpha La^2}{\rho^2+a^2}.$$

But in the limit that $\rho\to0$, this doesn't vanish, but takes the limiting value $q=4\pi\alpha L$. This implies a linear charge density

$$\lambda=\frac{q}{L}=4\pi\alpha.$$

2. The potential on a sphere of radius *a* is given by $\Phi(a,\theta,\phi)=V\cos\theta$. The potential on a sphere of radius 2*a* is given by $\Phi(2a,\theta,\phi)=V$. Find the potential in the region $a<r<2a$ assuming there are no charges between the two spheres.

We need to write each of these potentials in terms of spherical harmonics. Using the explicit form of the spherical harmonics, it is not hard to see that

$$\Phi(a,\theta,\phi)=V\sqrt{\tfrac{4}{3}\pi}Y_{10}(\theta,\phi),\quad \Phi(2a,\theta,\phi)=V\sqrt{4\pi}Y_{00}(\theta,\phi).$$

Now, the general solution of $\nabla^2\Phi=0$ in spherical coordinates is given by

$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^{l}+Br^{-l-1}\right)Y_{lm}(\theta,\phi).$$

By comparison with the given expressions, it seems to make sense to conjecture that only the values $(l,m)=(0,0)$ and $(l,m)=(1,0)$ will contribute. We therefore have

$$\Phi(r,\theta,\phi)=\left(A_{00}+B_{00}r^{-1}\right)Y_{00}(\theta,\phi)+\left(A_{10}r+B_{10}r^{-2}\right)Y_{00}(\theta,\phi).$$

We want the first pair of terms to vanish at $r = a$ and take the value $V\sqrt{4\pi}Y_{00}$ at $r = 2a$. This yields the two equations

$$A_{00}+B_{00}a^{-1}=0,\quad A_{00}+B_{00}(2a)^{-1}=V\sqrt{4\pi}$$

The first of these equations implies $B_{00}=-aA_{00}$, and substituting this into the other, we have

$$A_{00}-aA_{00}(2a)^{-1}=V\sqrt{4\pi},$$
$$\tfrac{1}{2}A_{00}=V\sqrt{4\pi},$$
$$A_{00}=2V\sqrt{4\pi}\quad\text{and}\quad B_{00}=-2aV\sqrt{4\pi}.$$

We also want the expression $A_{10}r+B_{10}r^{-2}$ to vanish at $r = 2a$, and take the value $V\sqrt{\tfrac{4}{3}\pi}$ at $r = a$. This yields the two equations

$$A_{10}a+B_{10}a^{-2}=V\sqrt{\tfrac{4}{3}\pi},\quad A_{10}(2a)+B_{10}(2a)^{-2}=0.$$

The second of these equations implies $B_{10}=-8a^3A_{10}$. Substituting into the first, we have

$$A_{10}a-8a^3A_{10}a^{-2}=V\sqrt{\tfrac{4}{3}\pi},$$
$$-7aA_{10}=V\sqrt{\tfrac{4}{3}\pi},$$
$$A_{10}=-\tfrac{1}{7}Va^{-1}\sqrt{\tfrac{4}{3}\pi},\quad B_{10}=\tfrac{8}{7}Va^{2}\sqrt{\tfrac{4}{3}\pi}.$$

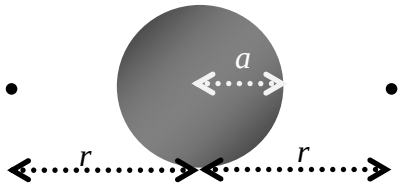
If we substitute these back into our expression for the potential, we have

$$\Phi(r,\theta,\phi)=2V\sqrt{4\pi}\left(1-\frac{a}{r}\right)Y_{00}(\theta,\phi)+\tfrac{1}{7}V\sqrt{\tfrac{4}{3}\pi}\left(-\frac{r}{a}+\frac{8a^2}{r^2}\right)Y_{10}(\theta,\phi).$$

We can then substitute in the explicit forms for the spherical harmonics to write our answer as

$$\Phi(r,\theta,\phi) = 2V\left(1-\frac{a}{r}\right)+\tfrac{1}{7}V\left(\frac{8a^2}{r^2}-\frac{r}{a}\right)\cos\theta.$$

3. Two matching point charges *q* are placed on opposite sides at a distance *r* from the center of a conducting sphere of radius *a*. Find the total force on one of the charges if the sphere is (a) grounded (b) neutral.



This calculation must be done by the method of images. For a grounded conducting sphere,there will be two point image charges of magnitude $q' = -qa/r$ at a distance of $r' = a^2/r$ from the center. The charges themselves do not provide any force on themselves, but they DO feel the force from the other charge. The distance between the two charges is $2r$, and the distance from the charge to the two image charges is $r \pm a^2/r$. The total force on the charge on the right will then be

$$F = \frac{\hat{\mathbf{x}}}{4\pi\varepsilon_0}\left[\frac{q^2}{4r^2}+\frac{qq'}{\left(r+a^2/r\right)^2}+\frac{qq'}{\left(r-a^2/r\right)^2}\right] = \frac{q^2\hat{\mathbf{x}}}{4\pi\varepsilon_0}\left[\frac{1}{4r^2}-\frac{a}{r\left(r+a^2/r\right)^2}-\frac{a}{r\left(r-a^2/r\right)^2}\right]$$

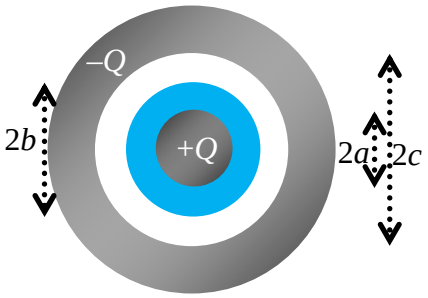
$$= \frac{q^2\hat{\mathbf{x}}}{4\pi\varepsilon_0}\left[\frac{1}{4r^2}-\frac{ar}{\left(r^2+a^2\right)^2}-\frac{ar}{\left(r^2-a^2\right)^2}\right]$$

If you insist that the sphere be neutral, you must add an additional charge $-2q'$ right at the center. We will therefore have

$$F = \frac{\hat{\mathbf{x}}}{4\pi\varepsilon_0}\left[\frac{q^2}{4r^2}+\frac{qq'}{\left(r+a^2/r\right)^2}+\frac{qq'}{\left(r-a^2/r\right)^2}-\frac{2qq'}{r^2}\right]$$

$$= \frac{q^2\hat{\mathbf{x}}}{4\pi\varepsilon_0}\left[\frac{1}{4r^2}-\frac{a}{r\left(r+a^2/r\right)^2}-\frac{a}{r\left(r-a^2/r\right)^2}+\frac{2a}{rr^2}\right] = \frac{q^2\hat{\mathbf{x}}}{4\pi\varepsilon_0}\left[\frac{1}{4r^2}-\frac{ar}{\left(r^2+a^2\right)^2}-\frac{ar}{\left(r^2-a^2\right)^2}+\frac{2a}{r^3}\right]$$

4. A solid conducting sphere of radius *a* is surrounded by a hollow conducting sphere of radius *c*. A charge *Q* is placed on the inner sphere and *–Q* on the outer sphere. The region from *r = a* to *r = b* is filled with an insulator with dielectric constant *ε*, and the rest is left in vacuum. Find D and E everywhere between the two spheres, and the potential difference $\Delta\Phi = \Phi_c - \Phi_a$. Also find the bound surface charge density at *r = b*, and the total energy.



The problem clearly has spherical symmetry, so it makes a lot of sense to assume that all fields will be strictly radial and depend only on the distance from the center. We therefore have, for example, $\mathbf{D} = \hat{\mathbf{r}} D(r)$. We then use Gauss's on a sphere of radius *r* anywhere in the region $a < r < c$ to see that

$$Q = \int_S \mathbf{D}\cdot\hat{\mathbf{n}}\, da = D(r)\int_S da = 4\pi r^2 D(r),$$

$$\mathrm{D}(r) = \frac{Q}{4\pi r^2}, \quad \text{so} \quad \mathbf{D}(\mathbf{x}) = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}.$$

If we divide this by ε in the insulating medium and by ε_0 in vacuum, we find the electric field as well.

$$\mathbf{E}(\mathbf{x}) = \frac{1}{\varepsilon(\mathbf{x})}\mathbf{D}(\mathbf{x}) = \frac{Q\hat{\mathbf{r}}}{4\pi r^2}\begin{cases}\varepsilon^{-1} & a < r < b,\\ \varepsilon_0^{-1} & b < r < c.\end{cases}$$

The potential difference is the integral of the electric field, so

$$\Delta\Phi = \Phi(a) - \Phi(c) = -\int_a^c (\nabla\Phi)\cdot d\mathbf{l} = \int_a^c E_r dr = \frac{Q}{4\pi}\left(\int_a^b \frac{dr}{\varepsilon r^2} + \int_b^c \frac{dr}{\varepsilon_0 r^2}\right) = \frac{Q}{4\pi}\left(\frac{-1}{r\varepsilon}\Big|_a^b + \frac{-1}{r\varepsilon_0}\Big|_b^c\right)$$

$$= \frac{Q}{4\pi}\left(\frac{1}{a\varepsilon} - \frac{1}{b\varepsilon} + \frac{1}{b\varepsilon_0} - \frac{1}{c\varepsilon_0}\right).$$

I unintentionally asked for the negative of this, which can be fixed by adding a minus sign.

The charge density on the surface is given by

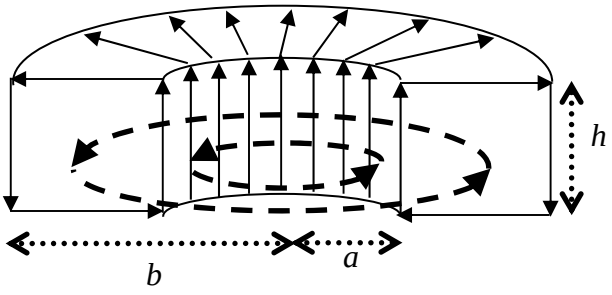
$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = (\mathbf{D} - \varepsilon_0\mathbf{E})\cdot\hat{\mathbf{r}} = \frac{Q}{4\pi b^2}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right).$$

The energy can then be computed using

$$W = \int w\, d^3\mathbf{x} = \tfrac{1}{2}\int \mathbf{E}\cdot\mathbf{D}\, d^3\mathbf{x} = \tfrac{1}{2}\int d\Omega\left[\int_a^b \frac{1}{\varepsilon}\mathbf{D}^2 r^2 dr + \int_b^c \frac{1}{\varepsilon_0}\mathbf{D}^2 r^2 dr\right] = \frac{2\pi}{(4\pi)^2}\left[\frac{1}{\varepsilon}\int_a^b \frac{dr}{r^2} + \frac{1}{\varepsilon_0}\int_b^c \frac{dr}{r^2}\right]$$

$$= \frac{1}{8\pi}\left(-\frac{1}{\varepsilon r}\Big|_a^b - \frac{1}{\varepsilon_0 r}\Big|_b^c\right) = \frac{1}{8\pi}\left(\frac{1}{\varepsilon a} - \frac{1}{\varepsilon b} + \frac{1}{\varepsilon_0 b} - \frac{1}{\varepsilon_0 c}\right).$$

5. A tokamak is in the shape of a rectangular cross-section donut centered on the *z*-axis, with height *h*, inner radius *a* and outer radius *b*, as sketched in the cutaway view at right. A total current *I* is then sent around the rectangular direction of the tokamak, so it goes up through the hole in the center, across the top, down on the outside, and then back to the center, equally at all angles, so as to generate a magnetic flux density $\mathbf{B}(\mathbf{x}) = B(\rho, z)\hat{\boldsymbol{\varphi}}$ in cylindrical coordinates. Find B at all points inside or outside the tokamak, and find the total magnetic energy stored inside the tokamak.



We are in vacuum, for which $\mathbf{B} = \mu_0\mathbf{H}$, or $\mathbf{B} = \mu_0\mathbf{H}$. We will use the integral version of Ampere's Law, which says that $I_S = \oint \mathbf{H}(\mathbf{x}) \cdot d\mathbf{l}$. We will use loops that go around in the $\hat{\boldsymbol{\varphi}}$ direction, such as the two dashed loops sketched in above, while remaining at constant ρ and *z*. Such a loop has a length of $2\pi\rho$, and therefore we have

$$\mu_0 I_S = \mu_0 \oint \mathbf{H}(\mathbf{x}) \cdot d\mathbf{l} = \oint \mathbf{B}(\mathbf{x}) \cdot d\mathbf{l} = 2\pi\rho B(\rho, z),$$

$$\mathbf{B}(\rho, z) = \frac{\mu_0 I_S \hat{\boldsymbol{\varphi}}}{2\pi\rho}.$$

where I_S is the current passing through an imaginary surface bounded by these loops. For the smaller Ampere loop, which lies within the "hole" of the donut, there is not current passing through it, and therefore we simply find $\mathbf{B} = 0$. Indeed, the same is true of any loop that is not within the "cake" of the donut, whether it be above, below, around the exterior, or in the hole. In contrast for any loop that is in the "cake" of the donut, the entire current *I* passes through the Ampere loop, so we have $I_s = I$. We therefore have, in summary

$$\mathbf{B}_{\text{in}} = \frac{\mu_0 I \hat{\boldsymbol{\varphi}}}{2\pi\rho}, \quad \mathbf{B}_{\text{out}} = 0.$$

The energy is then easily computed using

$$W = \int w\, d^3\mathbf{x} = \tfrac{1}{2}\int \mathbf{B}\cdot\mathbf{H}\, d^3\mathbf{x} = \frac{1}{2\mu_0}\int \mathbf{B}^2\, d^3\mathbf{x} = \frac{\mu_0 I^2}{8\pi^2}\int_{\text{in}} \frac{d^3\mathbf{x}}{\rho^2}$$

$$= \frac{\mu_0 I^2}{8\pi^2}\int_0^{2\pi} d\phi \int_a^b \frac{\rho\, d\rho}{\rho^2}\int_0^h dz = \frac{2\pi\mu_0 I^2 h}{8\pi^2}\ln(\rho)\Big|_a^b = \frac{1}{4\pi}\mu_0 I^2 h \ln\left(\frac{b}{a}\right).$$