Physics 712
Chapter X Problems

2. Consider a model in which electrons have a number density n_e and are in a damped harmonic oscillator, such that their displacement x in the presence of an electric field will be governed by $m\ddot{\mathbf{x}} = -e\mathbf{E} - m\gamma\dot{\mathbf{x}} - m\omega_0^2\mathbf{x}$. Assuming the positions and electric field are both proportional to $e^{-i\omega t}$, find the relationship between E and x. Then find the polarization $\mathbf{P} = -n_e e\mathbf{x}$, and the complex permittivity ε. As a check, make sure you get the same answer as we did for a collisionless plasma ($\gamma = \omega_0 = 0$).

Rewriting $\mathbf{x}(t) = \mathbf{x}e^{-i\omega t}$ and $\mathbf{E}(t) = \mathbf{E}e^{-i\omega t}$, we see that all time derivatives just become factors of $-i\omega$, so we have

$$-m\omega^2\mathbf{x} = -e\mathbf{E} + im\gamma\omega\mathbf{x} - m\omega_0^2\mathbf{x}$$

Solving for **x**, we have

$$m\omega_0^2\mathbf{x} - m\omega^2\mathbf{x} - im\gamma\omega\mathbf{x} = \frac{-e\mathbf{E}}{m\left(\omega_0^2 - \omega^2 - im\gamma\omega\right)}$$

The polarization will just be $\mathbf{P} = -ne\mathbf{x}$, and therefore

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0\mathbf{E} + \frac{ne^2\mathbf{E}}{m\left(\omega_0^2 - \omega^2 - im\gamma\omega\right)}.$$

Since $\mathbf{D} = \varepsilon\mathbf{E}$, the permittivity is therefore

$$\varepsilon = \varepsilon_0 + \frac{ne^2}{m\left(\omega_0^2 - \omega^2 - i\gamma\omega\right)}$$

If we compare this with the formula for a plasma, we have

$$\varepsilon = \varepsilon_0 + \frac{i\sigma}{\omega} = \varepsilon_0 - \frac{n_e e^2}{m\omega^2}.$$

This is clearly the same equation if we let $\gamma = \omega_0 = 0$.

3. What is the real part of the permittivity of a material if

$$\operatorname{Im}\left[\varepsilon(\omega)\right] = \begin{cases} a\left(\omega_0^2\omega - \omega^3\right) & \text{for } \omega < \omega_0 , \\ 0 & \text{for } \omega > \omega_0 ? \end{cases}$$

It should be noted that you will not have to use the principal part when attempting to find $\operatorname{Re}\left[\varepsilon(\omega)\right]$ **if** $\omega > \omega_0$**.**

We simply use the Kramers-Kronig relationship

$$\begin{aligned} \operatorname{Re}\left[\varepsilon(\omega)\right] &= \varepsilon_0 + \frac{2}{\pi} P\int_0^\infty \frac{\omega' \operatorname{Im}\left[\varepsilon(\omega')\right]}{\omega'^2 - \omega^2} d\omega' = \varepsilon_0 + \frac{2a}{\pi} P\int_0^{\omega_0} \frac{\omega'^2\left(\omega_0^2 - \omega'^2\right)}{\omega'^2 - \omega^2} d\omega' \\ &= \varepsilon_0 + \frac{2a}{\pi} P\int_0^{\omega_0} \left[-\omega'^2 + \left(\omega_0^2 - \omega^2\right)\left(1 - \frac{\frac{1}{2}\omega}{\omega' + \omega} + \frac{\frac{1}{2}\omega}{\omega' - \omega}\right)\right] d\omega' \\ &= \varepsilon_0 + \frac{2a}{\pi} P\left\{-\tfrac{1}{3}\omega'^3 + \left(\omega_0^2 - \omega^2\right)\left[\omega' - \tfrac{1}{2}\omega \ln\left(\omega' + \omega\right) + \tfrac{1}{2}\omega \ln\left|\omega' - \omega\right|\right]\right\}\Big|_0^{\omega_0} . \end{aligned}$$

Only the final term requires any care in evaluating the limit, as we must avoid the pole at $\omega' = \omega$. When $\omega > \omega_0$, there is no pole, and this last term just yields $\ln\left(\omega - \omega_0\right) - \ln\left(\omega_0\right)$, but if $\omega < \omega_0$, then

$$\begin{aligned} P\left(\ln\left|\omega' - \omega\right|\right)\Big|_0^{\omega_0} &= \lim_{\delta \to 0^+} \left[\left(\ln\left|\omega' - \omega\right|\right)\Big|_0^{\omega - \delta} + \left(\ln\left|\omega' - \omega\right|\right)\Big|_{\omega + \delta}^{\omega_0}\right] \\ &= \lim_{\delta \to 0^+} \left[\ln\left|-\delta\right| - \ln\left|-\omega\right| + \ln\left|\omega_0 - \omega\right| - \ln\left|\delta\right|\right] = \ln\left(\frac{\left|\omega_0 - \omega\right|}{\omega}\right). \end{aligned}$$

The last expression works in both cases. So we have

$$\begin{aligned} \operatorname{Re}\left[\varepsilon(\omega)\right] &= \varepsilon_0 + \frac{2a}{\pi}\left\{\frac{2}{3}\omega_0^3 - \omega_0\omega^2 + \frac{1}{2}\omega\left(\omega_0^2 - \omega^2\right)\left[\ln\left(\frac{\left|\omega - \omega_0\right|}{\omega}\right) - \ln\left(\frac{\omega + \omega_0}{\omega}\right)\right]\right\} \\ &= \varepsilon_0 + \frac{2a}{\pi}\left[\frac{2}{3}\omega_0^3 - \omega_0\omega^2 + \frac{1}{2}\omega\left(\omega_0^2 - \omega^2\right)\ln\left(\frac{\left|\omega - \omega_0\right|}{\omega + \omega_0}\right)\right] \end{aligned}$$