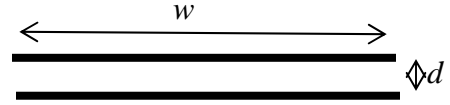


Physics 712 Chapter 8 Solutions

1. Consider a wave guide consisting of two long parallel flat conductors, separated by a distance d in the y -direction, and having width w in the x -direction, with $w \gg d$ (ignore edge effects). Find general formulas for the electric and magnetic field for waves moving in the $+z$ direction, if the electric field has magnitude E_0 . Find the time-averaged Poynting vector $\langle \mathbf{S} \rangle$ and integrate it over the space between the conductors to find the rate of energy transmission.



For TEM modes, the electric field can be derived from a potential whose value is constant on each of these two conductors. In the limit $w \gg d$, this produces a uniform field which points from one side to the other, so we simply have $\mathbf{E}_t = E_0 \hat{\mathbf{y}}$, or including the z -dependence, $\mathbf{E} = E_0 e^{ikz} \hat{\mathbf{y}}$. The magnetic field is then given by $\mathbf{B}_t = \sqrt{\epsilon\mu} \hat{\mathbf{z}} \times \mathbf{E}_t$ for waves in the z -direction, which means $\mathbf{B} = -\sqrt{\epsilon\mu} E_0 e^{ikz} \hat{\mathbf{x}}$. Each of these should then be multiplied by $e^{-i\omega t}$ and the real part taken to get the actual space and time dependence.

The resulting Poynting vector is given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{1}{4} (\mathbf{E} \times \mathbf{H} e^{-2i\omega t} + \mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H} + \mathbf{E}^* \times \mathbf{H}^* e^{2i\omega t}).$$

When we time average, the factors of $e^{\pm 2i\omega t}$ average to zero to yield the usual expression,

$$\langle \mathbf{S} \rangle = \frac{1}{4} (\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H}) = \frac{1}{2} \text{Re} (\mathbf{E} \times \mathbf{H}^*) = \frac{1}{2} \mu^{-1} \text{Re} (\mathbf{E} \times \mathbf{B}^*)$$

In this case, this yields

$$\langle \mathbf{S} \rangle = -\mu^{-1} \sqrt{\epsilon\mu} \frac{1}{2} \text{Re} (E_0 e^{ikz} \hat{\mathbf{y}} \times E_0^* e^{-ikz} \hat{\mathbf{x}}) = \frac{1}{2} \sqrt{\epsilon/\mu} |E_0|^2 \hat{\mathbf{z}}$$

If we integrate this over the region between the conductors, we get a total power

$$P = \int \langle \mathbf{S} \rangle \cdot \hat{\mathbf{z}} da = \frac{1}{2} wd \sqrt{\epsilon/\mu} |E_0|^2.$$

2. Electrical power is transmitted by the use of wave guides using TEM modes, with one conductor being the wire and the other being the Earth. We will model this as a coaxial cable with wire radius $a = 1.0$ cm and “Earth distance” $b = 10.0$ m, with air (vacuum) in between.

(a) For TEM modes of the form given in class, if it is travelling in the $+z$ direction, find the time-averaged Poynting vector $\langle \mathbf{S} \rangle$. Integrate it over the whole area between the conductors to get the rate of power transmission.

The electric and magnetic fields were already worked out, so we write them in the form

$$\mathbf{E} = \frac{C}{\rho} \hat{\rho} e^{ikz - i\omega t}, \quad \mathbf{B} = \frac{C\sqrt{\epsilon_0\mu_0}}{\rho} \hat{\phi} e^{ikz - i\omega t}$$

We then get the time averaged Poynting vector using the same equation as before, namely

$$\langle \mathbf{S} \rangle = \frac{1}{2} \text{Re}(\mathbf{E} \times \mathbf{H}^*) = \frac{1}{2\mu_0} \left[\frac{C}{\rho} \hat{\rho} e^{ikz - i\omega t} \right] \times \left[\frac{C\sqrt{\epsilon_0\mu_0}}{\rho} \hat{\phi} e^{-ikz + i\omega t} \right] = \frac{C^2}{2\rho^2} \sqrt{\frac{\epsilon_0}{\mu_0}} \hat{\mathbf{z}}$$

To get the power, we integrate over the region between the cables, which implies

$$P = \int \langle \mathbf{S} \rangle \cdot \hat{\mathbf{z}} da = \frac{C^2}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} \int_0^{2\pi} d\phi \int_a^b \frac{\rho d\rho}{\rho^2} = \pi C^2 \sqrt{\epsilon_0/\mu_0} \ln(b/a)$$

(b) If the power transmitted is 1.00 MW, determine the voltage difference between the inner and outer conductors.

The voltage difference between the two conductors will be

$$\Delta\Phi = \Phi(b) - \Phi(a) = -\int_a^b \mathbf{E} \cdot \hat{\rho} d\rho = -C e^{ikz - i\omega t} \int_a^b \frac{d\rho}{\rho} = -C e^{ikz - i\omega t} \ln(b/a)$$

The voltage difference is a function of time and space (not surprising for a traveling wave), but the *magnitude* of this difference is just given by $\Delta\Phi = C \ln(b/a)$. Solving for C , we can then see that

$$P = \pi C^2 \sqrt{\epsilon_0/\mu_0} \ln(b/a) = \pi \left[\frac{\Delta\Phi}{\ln(b/a)} \right]^2 \sqrt{\epsilon_0/\mu_0} \ln(b/a) = \frac{\pi (\Delta\Phi)^2}{\ln(b/a)} \sqrt{\frac{\epsilon_0}{\mu_0}}$$

We then solve this for the voltage difference and substitute in concreted numbers to find

$$\begin{aligned} (\Delta\Phi)^2 &= \frac{P \ln(b/a)}{\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{1}{\pi} (1.00 \times 10^6 \text{ J/s}) \ln(10^3) \sqrt{\frac{4\pi \times 10^{-7} \text{ N/A}^2}{8.854 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}}} \\ &= 8.28 \times 10^8 \text{ N m s}^{-1} \text{ N m C}^{-1} \text{ A}^{-1} = 8.28 \times 10^8 \text{ N}^2 \text{ m}^2 \text{ C}^{-2} = 8.28 \times 10^8 \text{ V}^2, \\ \Delta\Phi &= 28.8 \text{ kV}. \end{aligned}$$