

## Physics 712

### Chapter 7 Solutions

3. Some material are *birefringent*: they have different indices of refraction in two directions. Suppose a material has index of refraction  $n_x$  for electric fields in the  $x$ -direction and index of refraction  $n_y$  in the  $y$ -direction, with  $n_y > n_x$ . Now imagine a wave going through such a region of thickness  $d$  which has initial polarization  $\boldsymbol{\varepsilon} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + \hat{\mathbf{y}})$ . Show there is a minimum distance  $d$  such that the wave will now be circularly polarized. A device with this thickness is called a *quarter wave plate*. Then tell me what happens if the same initial wave were passed through two, three, or four quarter-wave plates. What would happen if the same initial wave were put through two, three, or four quarter-wave plates?

The electric field would normally take the form  $\mathbf{E}_0 \exp(ikz - i\omega t)$ , where  $kc = n\omega$ , or  $k = n\omega/c$ . However, in this case, we have  $k_x = n_x\omega/c$  and  $k_y = n_y\omega/c$ . If it starts at  $z = 0$  with an electric field  $\mathbf{E} = E_0\boldsymbol{\varepsilon}e^{-i\omega t} = \frac{1}{\sqrt{2}}E_0(\hat{\mathbf{x}} + \hat{\mathbf{y}})e^{-i\omega t}$ , then at arbitrary  $z$  the wave will look like

$$\mathbf{E} = E_0\boldsymbol{\varepsilon}e^{-i\omega t} = \frac{1}{\sqrt{2}}E_0\left(\hat{\mathbf{x}}e^{ik_x z - i\omega t} + \hat{\mathbf{y}}e^{ik_y z - i\omega t}\right) = \frac{1}{\sqrt{2}}E_0\left(\hat{\mathbf{x}} + e^{i(k_y - k_x)z}\hat{\mathbf{y}}\right)e^{ik_x z - i\omega t}$$

The final factor is just a phase factor. The polarization is now

$$\boldsymbol{\varepsilon} = \frac{1}{\sqrt{2}}\left[\hat{\mathbf{x}} + e^{i(k_y - k_x)z}\hat{\mathbf{y}}\right].$$

Now, circularly polarized light (with positive helicity) would have  $\boldsymbol{\varepsilon} = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + i\hat{\mathbf{y}})$ , so if we let  $(k_y - k_x)d = \frac{1}{2}\pi$ , we would have  $e^{i(k_y - k_x)d} = e^{i\pi/2} = i$ , and we would get such light. The minimum distance is therefore

$$d = \frac{\pi}{2(k_y - k_x)} = \frac{\pi c}{2\omega(n_y - n_x)}.$$

For two, three, or four quarter wave plates, we would now find

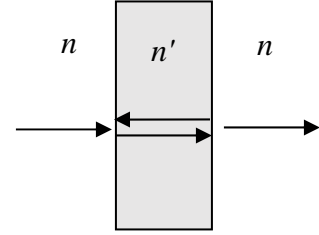
$$\text{two: } \boldsymbol{\varepsilon} = \frac{1}{\sqrt{2}}\left[\hat{\mathbf{x}} + e^{i(k_y - k_x)2d}\hat{\mathbf{y}}\right] = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{x}} + e^{i\pi}\hat{\mathbf{y}}\right) = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - \hat{\mathbf{y}}),$$

$$\text{three: } \boldsymbol{\varepsilon} = \frac{1}{\sqrt{2}}\left[\hat{\mathbf{x}} + e^{i(k_y - k_x)3d}\hat{\mathbf{y}}\right] = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{x}} + e^{3i\pi/2}\hat{\mathbf{y}}\right) = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - i\hat{\mathbf{y}}),$$

$$\text{four: } \boldsymbol{\varepsilon} = \frac{1}{\sqrt{2}}\left[\hat{\mathbf{x}} + e^{i(k_y - k_x)4d}\hat{\mathbf{y}}\right] = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{x}} + e^{2i\pi}\hat{\mathbf{y}}\right) = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + \hat{\mathbf{y}}).$$

For two it is linearly polarized, but now the other way, for three it is negative helicity polarization, and for four it is back to the original linear polarization.

4. A plane wave starts in a region with index of refraction  $n$  and then is normally incident on a region with index  $n'$  of thickness  $d$ , after which it then exits to a region of index  $n$ . For what thicknesses  $d$  will there be *no* reflected wave? Will the same thickness  $d$  work for all frequencies? Explain why soap bubbles often look colorful in reflected light.



Following the notes, we will have the wave propagating in the  $z$ -direction, and the index  $n'$  region will go from  $z = 0$  to  $z = d$ . Then we will have waves going to the right on the right and left, but waves going both directions in the middle, so we have

$$\mathbf{E} = E\hat{\mathbf{x}}e^{ikz-i\omega t}, \quad \mathbf{E}' = E_1\hat{\mathbf{x}}e^{ik'z-i\omega t} + E_2\hat{\mathbf{x}}e^{-ik'z-i\omega t}, \quad \mathbf{E}'' = E''\hat{\mathbf{x}}e^{ikz-i\omega t}.$$

The magnetic field is found from  $\mathbf{E} = n\hat{\mathbf{k}} \times \mathbf{E}/c$ , so as in the notes, these are given by

$$\mathbf{B} = nE\hat{\mathbf{y}}e^{ikz-i\omega t}/c, \quad \mathbf{B}' = n'E_1\hat{\mathbf{y}}e^{ik'z-i\omega t}/c - n'E_2\hat{\mathbf{y}}e^{-ik'z-i\omega t}/c, \quad \mathbf{B}'' = nE''\hat{\mathbf{y}}e^{ikz-i\omega t}/c$$

We must match both the electric and magnetic fields at the boundaries (normally, you would match  $D_z$ , but there is no  $D_z$ ). We therefore have

$$E = E_1 + E_2, \quad E_1e^{ik'd} + E_2e^{-ik'd} = E''e^{ikd}, \quad nE = n'E_1 - n'E_2, \quad n'E_1e^{ik'd} - n'E_2e^{-ik'd} = nE''e^{ikd}.$$

Substituting the first equation in the third and the second in the fourth, we have

$$nE_1 + nE_2 = n'E_1 - n'E_2, \quad n'E_1e^{ik'd} - n'E_2e^{-ik'd} = nE_1e^{ik'd} + nE_2e^{-ik'd}.$$

Place  $E_1$  on one side of the equation and  $E_2$  on the other

$$(n' + n)E_2 = (n' - n)E_1, \quad (n' - n)E_1e^{ik'd} = (n' + n)E_2e^{-ik'd}.$$

Substituting the first equation into the second to eliminate  $E_2$ , we have

$$(n' - n)E_1e^{ik'd} = (n' - n)E_1e^{-ik'd}, \\ e^{ik'd} = e^{-ik'd}, \\ e^{2ik'd} = 1.$$

This works whenever  $2k'd = 2\pi n$ , or  $d = \frac{\pi n}{k'}$ . This means that  $d$  must be an integer or half-integer number of wavelengths thick.

Of course, in general, the wavelength will depend on frequency, and hence what works for one wavelength will not work for others. This means that if you have polychromatic light reflecting off a very thin layer with a different index of refraction (like the wall of a soap bubble), some colors will interfere destructively, as we've found here, while others will not. Hence some colors are reflected and some aren't. Hence white light, when reflected from such a thin layer, will look colorful.