Physics 712
Chapter 6 Solutions

1. A wire along the *z*-axis has a current that turns on suddenly at *t* = 0, so the current density is $\mathbf{J}(\mathbf{x},t) = I\hat{\mathbf{z}}\delta(x)\delta(y)\theta(t)$, where $\theta(t)$ is the Heaviside function, with $\theta(t<0)=0$ and $\theta(t>0)=1$. There is no charge density, $\rho(\mathbf{x},t)=0$.

(a) Working in Lorentz gauge, find $\mathbf{A}(\mathbf{x},t)$ in cylindrical coordinates.

Whether we work in Coulomb or Lorentz gauge, there is no scalar potential, so $\Phi(\mathbf{x},t)=0$. The vector potential is given by

$$\mathbf{A}(\mathbf{x},t) = \mu_0 \int \frac{\mathbf{J}\left(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c\right)}{4\pi|\mathbf{x}-\mathbf{x}'|} d^3\mathbf{x}' = \frac{\mu_0 I}{4\pi}\hat{\mathbf{z}}\int \frac{\delta(x')\delta(y')\theta\left(t-|\mathbf{x}-\mathbf{x}'|/c\right)}{|\mathbf{x}-\mathbf{x}'|} d^3\mathbf{x}'$$

$$= \frac{\mu_0 I}{4\pi}\hat{\mathbf{z}}\int_{-\infty}^{\infty} \frac{\theta\left(t-|\mathbf{x}-z'\hat{\mathbf{z}}|/c\right)dz'}{|\mathbf{x}-z'\hat{\mathbf{z}}|} = \frac{\mu_0 I}{4\pi}\hat{\mathbf{z}}\int_{-\infty}^{\infty} \frac{\theta\left(t-\sqrt{x^2+y^2+(z-z')^2}\,/c\right)dz'}{\sqrt{x^2+y^2+(z-z')^2}}$$

We now switch to cylindrical coordinates, so that $x^2+y^2=\rho^2$. Although the integral can be completed using a trigonometric substitution, it is actually easier to make the hyperbolic substitution $z' = z + \rho\sinh\psi$. We then have

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0 I}{4\pi}\hat{\mathbf{z}}\int_{-\infty}^{\infty} \frac{\theta\left(t-\sqrt{\rho^2+\rho^2\sinh^2\psi}\,/c\right)}{\sqrt{\rho^2+\rho^2\sinh^2\psi}}\rho\cosh\psi\, d\psi = \frac{\mu_0 I}{4\pi}\hat{\mathbf{z}}\int_{-\infty}^{\infty}\theta\left(t-\frac{\rho\cosh\psi}{c}\right)d\psi$$

The Heaviside function vanishes unless $\rho\cosh\psi > ct$. If $\rho < ct$, this can never happen, but if $\rho > ct$ it works provided $|\psi| < \cosh^{-1}(ct/\rho)$, so we have

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0 I}{4\pi}\hat{\mathbf{z}}\int_{-\cosh^{-1}(ct/\rho)}^{\cosh^{-1}(ct/\rho)} d\psi = \frac{\mu_0 I}{2\pi}\cosh^{-1}(ct/\rho)\,\hat{\mathbf{z}}.$$

(b) Find the electric and magnetic fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$.

For the electric and magnetic fields, we use $\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla\times\mathbf{A}$ to obtain

$$\mathbf{E} = -\nabla\Phi - \frac{\partial}{\partial t}\mathbf{A} = -\frac{\mu_0 I}{2\pi}\frac{\partial}{\partial t}\cosh^{-1}(ct/\rho)\,\hat{\mathbf{z}} = -\frac{\mu_0 I}{2\pi}\hat{\mathbf{z}}\frac{1}{\sqrt{c^2t^2/\rho^2-1}}\frac{c}{\rho} = -\frac{\mu_0 cI}{2\pi\sqrt{c^2t^2-\rho^2}}\hat{\mathbf{z}},$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{\mu_0 I}{2\pi}\left[\frac{\hat{\boldsymbol{\rho}}}{\rho}\frac{\partial}{\partial\phi}A_z - \hat{\boldsymbol{\varphi}}\frac{\partial}{\partial\rho}A_z\right] = -\frac{\mu_0 I}{2\pi}\hat{\boldsymbol{\varphi}}\frac{\partial}{\partial t}\cosh^{-1}(ct/\rho) = -\frac{\mu_0 I}{2\pi}\hat{\boldsymbol{\varphi}}\frac{1}{\sqrt{c^2t^2/\rho^2-1}}\frac{-ct}{\rho^2},$$

$$\mathbf{B} = \frac{\mu_0 Ict}{2\pi\rho\sqrt{c^2t^2-\rho^2}}\hat{\boldsymbol{\varphi}}.$$