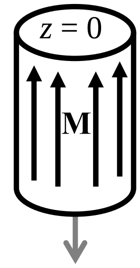
Physics 712
Chapter 5 Problems



6. [10] A hard magnet is in the shape of a semi-infinite cylinder of radius *R* with its main axis on the *z*-axis and running from *z* = – ∞ to *z* = 0. It has magnetization $\mathbf{M} = M\hat{\mathbf{z}}$ uniformly inside it. Find the surface current K on the surface, and then use the Biot-Savart law for current density to find B on every point on the *z*-axis.

The surface current is given by $\mathbf{K} = \mathbf{M}\times\hat{\mathbf{n}}$ on the surface. On the surface $z = 0$, this is zero, since both $\mathbf{M}$ and $\hat{\mathbf{n}}$ are in the z –direction. But if we work in cylindrical coordinates, then on the lateral surface, $\hat{\mathbf{n}} = \hat{\boldsymbol{\rho}}$, and hence we have $\mathbf{K} = \mathbf{M}\times\hat{\mathbf{n}} = M\hat{\mathbf{z}}\times\hat{\boldsymbol{\rho}} = M\hat{\boldsymbol{\varphi}}$. This means there is a surface current flowing counter-clockwise (as viewed from above) around the cylinder.

The Biot-Savart law (for current density) tells us the magnetic field is given by

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{x}')\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}d^3\mathbf{x}' = \frac{\mu_0}{4\pi}\int_S \mathbf{K}(\mathbf{x}')\times\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}da',$$

In this case, $\mathbf{x} = z\hat{\mathbf{z}}$ and $\mathbf{x}' = R\hat{\boldsymbol{\rho}} + z'\hat{\mathbf{z}}$. Substituting this into our expression, we have

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 M}{4\pi}\int_S \hat{\boldsymbol{\varphi}}\times\frac{-R\hat{\boldsymbol{\rho}}+(z-z')\hat{\mathbf{z}}}{\left[R^2+(z-z')^2\right]^{3/2}}da' = \frac{\mu_0 M}{4\pi}\int_S \frac{R\hat{\mathbf{z}}+(z-z')\hat{\boldsymbol{\rho}}}{\left[R^2+(z-z')^2\right]^{3/2}}da'$$

Now, this equation is a little confusing, because we have set it all up in cylindrical coordinates, The direction $\hat{\mathbf{z}}$ means the same thing everywhere, but the direction $\hat{\boldsymbol{\rho}}$ means something different depending on where we are on the surface of the cylinder. In fact, as we go around the surface of the cylinder, it is pretty clear that the $\hat{\boldsymbol{\rho}}$ direction averages to zero, and therefore, this term will integrate to zero. We therefore drop this term. For the da' integration we can think of this as $da' = Rd\phi'\,dz'$, so we find

$$\mathbf{B}(z) = \frac{\mu_0 MR^2\hat{\mathbf{z}}}{4\pi}\int_0^{2\pi}d\phi'\int_{-\infty}^0\frac{dz'}{\left[R^2+(z-z')^2\right]^{3/2}} = \left.\frac{\mu_0 M\hat{\mathbf{z}}(z'-z)}{2\sqrt{R^2+(z-z')^2}}\right|_{-\infty}^0 = \tfrac{1}{2}\mu_0 M\hat{\mathbf{z}}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)$$

7. [5] Repeat problem 6, but this time use the magnetic potential Φ_M approach. Find the magnetic potential on the *z*-axis, and then find the magnetic field H at all points on the *z*-axis. If you have been careful, you will find that $\mathbf{B} = \mu_0 \mathbf{H}$ for *z* > 0, but not for *z* < 0. Explain why.

Because the magnetization is constant inside the magnet, it satisfies $\nabla \cdot \mathbf{M} = 0$ in the interior, so there is only a contribution from the surface terms. The surface term only comes from the top, where we have $\mathbf{M} \cdot \hat{\mathbf{n}} = M$, so we have

$$\Phi_M(z) = \frac{1}{4\pi}\int_S \frac{\mathbf{M}(\mathbf{x}')\cdot\hat{\mathbf{n}}'}{|\mathbf{x}-\mathbf{x}'|}da' = \frac{M}{4\pi}\int_0^{2\pi} d\phi' \int_0^R \frac{\rho' d\rho'}{|z\hat{\mathbf{z}}-\rho'\hat{\boldsymbol{\rho}}|} = \frac{M}{2}\int_0^R \frac{\rho' d\rho'}{\sqrt{z^2+\rho'^2}} = \tfrac{1}{2}M\sqrt{z^2+\rho'^2}\Big|_0^R$$

$$= \tfrac{1}{2}M\left(\sqrt{z^2+R^2}-\sqrt{z^2}\right) = \tfrac{1}{2}M\left(\sqrt{z^2+R^2}-|z|\right).$$

The magnetic field is, therefore,

$$\mathbf{H} = -\nabla\Phi_M(z) = -\tfrac{1}{2}M\hat{\mathbf{z}}\frac{d}{dz}\left(\sqrt{z^2+R^2}-|z|\right) = \tfrac{1}{2}M\hat{\mathbf{z}}\left[\operatorname{sgn}(z) - \frac{z}{\sqrt{z^2+R^2}}\right],$$

where $\operatorname{sgn}(z)$ is defined to be +1 if $z > 0$ and –1 if $z < 0$. Multiplying by μ_0, we clearly get $\mathbf{B} = \mu_0\mathbf{H}$ for $z > 0$, but for $z < 0$ we get $\mathbf{B} - \mu_0\mathbf{H} = \mu_0 M\hat{\mathbf{z}}$. This is because $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$, and we have magnetization inside the magnet.