Physics 712
Chapter 5 Problems

4. [5] Consider a localized static current distribution. By considering the expression $\int \nabla \cdot \left[x_i x_j \mathbf{J}(\mathbf{x}) \right] d^3\mathbf{x}$**, show** $\int x_i J_j (\mathbf{x}) d^3\mathbf{x} = -\int x_j J_i (\mathbf{x}) d^3\mathbf{x}$**.**

We first use the divergence theorem to turn it into a surface integral, which then vanishes because the currents at infinity vanish:

$$\int \nabla \cdot \left[x_i x_j \mathbf{J}(\mathbf{x}) \right] d^3\mathbf{x} = \int_S x_i x_j \mathbf{J}(\mathbf{x}) \cdot \hat{\mathbf{n}} \, da = 0 .$$

We now distribute the gradient using the product rule, using the fact that a static current distribution has $\nabla \cdot \mathbf{J} = 0$, and then rewriting the dot products as sums to figure out what is going on.

$$0 = \int \nabla \cdot \left[x_i x_j \mathbf{J}(\mathbf{x}) \right] d^3\mathbf{x} = \int \left[x_j \mathbf{J}(\mathbf{x}) \cdot \nabla x_i + x_i \mathbf{J}(\mathbf{x}) \cdot \nabla x_j + x_i x_j \nabla \cdot \mathbf{J}(\mathbf{x}) \right] d^3\mathbf{x}$$

$$= \sum_k \int \left[x_j J_k (\mathbf{x}) \frac{\partial x_i}{\partial x_k} + x_i J_k (\mathbf{x}) \frac{\partial x_j}{\partial x_k} \right] d^3\mathbf{x} = \sum_k \int \left[x_j J_k (\mathbf{x}) \delta_{ik} + x_i J_k (\mathbf{x}) \delta_{jk} \right] d^3\mathbf{x}$$

$$= \int \left[x_j J_i (\mathbf{x}) + x_i J_j (\mathbf{x}) \right] d^3\mathbf{x} = \int x_j J_i (\mathbf{x}) d^3\mathbf{x} + \int x_i J_j (\mathbf{x}) d^3\mathbf{x} .$$

The desired identity is now a trivial step away.

5. [5] For each of the four loops in problem 2, find the magnetic dipole moment m and the resulting magnetic flux density far from the loop.

The magnetic dipole moment is just $\mathbf{m} = IA\hat{\mathbf{z}}$, because $\hat{\mathbf{z}}$ is the normal to the plane of the loop. The area is most easily computed by finding the area of one triangle and then multiplying them by the number of triangles. The area of one triangle is

$$A_{\text{one}} = \frac{1}{2}bd = \frac{1}{2}\frac{C}{N} \cdot \frac{C}{2N \tan (\pi/N)}$$

Multiplying by the number of triangles N, our answer is

$$\mathbf{m} = \frac{IC^2\hat{\mathbf{z}}}{4N \tan (\pi/N)}$$

Again, we can just substitute in our four values of N, using again $N \to \infty$ for the circle:

$$\mathbf{m}_3 = \frac{IC^2\hat{\mathbf{z}}}{12\sqrt{3}}, \quad \mathbf{m}_4 = \frac{IC^2\hat{\mathbf{z}}}{16}, \quad \mathbf{m}_6 = \frac{IC^2\hat{\mathbf{z}}}{8\sqrt{3}}, \quad \mathbf{m}_\infty = \frac{IC^2\hat{\mathbf{z}}}{4\pi} .$$

The magnetic flux density is just

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{3(\hat{\mathbf{r}} \cdot \mathbf{m})\hat{\mathbf{r}} - \mathbf{m}}{r^3} .$$