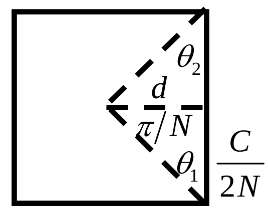
# Physics 712
# Chapter 5 Problems

**2. [10] A wire loop centered on the origin in the *xy*-plane has perimeter *C* and carries a current *I* traveling counter-clockwise as viewed from above. Find the magnetic flux density at the origin if the loop is a (a) equilateral triangle, (b) square, (c) regular hexagon, (d) circle.**




In every case except the last one, the sides of the regular polygon will be straight segments with length $L = C/N$. It is also not hard to figure out the distance of the segments to the origin. As you can see for the illustration at right, using the case $N = 4$ for illustrative purposes, the central angle from the origin to the closest point on the wire segment will be $2\pi/N$, and hence the central angle of half the side will be $\pi/N$. It is then clear that the distance $d$ to the origin is given by

$$\tan(\pi/N) = \frac{C/(2N)}{d},$$

$$d = \frac{C}{2N\tan(\pi/N)}.$$

The magnetic flux density from a line segment is given by

$$B_{\text{seg}} = \frac{\mu_0 I}{4\pi d}\left[\cos\theta_1 + \cos\theta_2\right]$$

where $\theta_1$ and $\theta_2$ are the base angles of the triangle formed by the line segment with lines going to the center. By the right-hand rule, this magnetic field will point in the $+z$ direction. By geometry, we can see that $\theta_1 = \theta_2 = \tfrac{1}{2}\pi - \tfrac{1}{N}\pi$, so that

$$\cos(\theta_1) = \cos(\theta_2) = \cos\left(\tfrac{1}{2}\pi - \tfrac{1}{N}\pi\right) = \sin(\pi/N)$$

We therefore have the magnetic field from one side of the polygon:

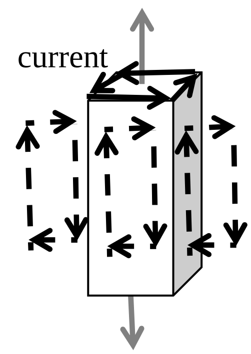
$$\mathbf{B}_{\text{seg}} = \frac{\mu_0 I}{2\pi d}\sin\left(\frac{\pi}{N}\right) = \frac{N\mu_0}{\pi C}\sin\left(\frac{\pi}{N}\right)\tan\left(\frac{\pi}{N}\right)$$

If we multiply this by the number of segments $N$, we get the magnetic field from the whole thing:

$$\mathbf{B}_N = \frac{N^2\mu_0 I}{\pi C}\sin\left(\frac{\pi}{N}\right)\tan\left(\frac{\pi}{N}\right)$$

We can then get each of our regular polygons simply by substituting the corresponding value of $N$. Indeed, we can even get a circle by considering the limit $N \to \infty$, in which case we use the approximation $\sin\theta \approx \tan\theta = \theta$ for small $\theta$.

$$\mathbf{B}_3 = \frac{27\mu_0 I}{2\pi C}\hat{\mathbf{z}}, \quad \mathbf{B}_4 = \frac{8\sqrt{2}\mu_0 I}{\pi C}\hat{\mathbf{z}}, \quad \mathbf{B}_6 = \frac{6\sqrt{3}\mu_0 I}{\pi C}\hat{\mathbf{z}}, \quad \mathbf{B}_\infty = \frac{\pi\mu_0 I}{C}\hat{\mathbf{z}}.$$




**3. [10] Consider a cylinder of arbitrary cross-sectional shape, such as a square, circle, or other similar shape. This cylinder will be infinitely long. It will have a surface current *K*, with units A/m, running around it in a counter-clockwise direction as viewed from above. Because it is infinitely long, you can use symmetry arguments to show that the magnetic flux density will always be parallel to the axis everywhere inside and outside the cylinder.**

**(a) [6] By considering the Ampere loop wholly inside the cylinder (middle dashed loop), argue that the magnetic field is in fact constant everywhere outside the cylinder. Repeat for the loop outside the cylinder (left dashed loop). If we assume the magnetic field at infinity is zero, what is the magnetic field everywhere outside this cylinder?**

Let the vertical size of any of the three Ampere's loops be *L*. By Ampere's law, since the loop on the inside has no current running through it, $\oint \mathbf{B}\cdot d\mathbf{l}=0$. Since the magnetic field points in the up direction (which I'll call +*z*), the top and bottom lines on that loop will have $\mathbf{B}\cdot d\mathbf{l}=0$, and will not contribute. On each of the two sides, the magnetic field cannot depend on *z*, and hence will be constant along either of the two sides of the loop. Hence we have

$$0=\oint \mathbf{B}\cdot d\mathbf{l}=\int_L \mathbf{B}\cdot d\mathbf{l}+\int_R \mathbf{B}\cdot d\mathbf{l}=B_L L-B_R L\ ,$$

where $B_R$ and $B_L$ are the magnetic flux density on the left and right side of the loop. Hence $B_R=B_L$, and since this loop can be found anywhere inside the cylinder, we have $\mathbf{B}_{in}=B_{in}\hat{\mathbf{z}}$, a constant, everywhere inside.

The argument is identical using the outside loop, and we conclude that $\mathbf{B}_{out}=B_{out}\hat{\mathbf{z}}$ everywhere outside. But since it vanishes at infinity (which is outside), we have $\mathbf{B}_{out}=0$.

**(b) [4] By considering the Ampere loop that is partly inside the cylinder and partly outside it (right dashed loop), find the magnetic flux density inside the cylinder.**

We once again use Ampere's Law, but if the vertical direction has length *L*, then we realize that a current *KL* is running through the loop, since it now passes into the cylinder. If we use Ampere's Law, we have $\oint \mathbf{B}\cdot d\mathbf{l}=\mu_0 I=\mu_0 KL$. But when we do the loop integral, the right side does not contribute (because $\mathbf{B}_{out}=0$), and the top and bottom don't contribute (because the flux density is always in the *z*-direction), so we have

$$\mu_0 KL=\oint \mathbf{B}\cdot d\mathbf{l}=\int_L \mathbf{B}\cdot d\mathbf{l}=B_{in}L\ ,$$

$$B_{in}=\mu_0 K\ .$$

Putting all our information together, we have

$$\mathbf{B}=\begin{cases}\mu_0 K\hat{\mathbf{z}} & \text{inside,}\\ 0 & \text{outside.}\end{cases}$$