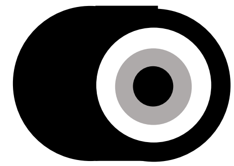
Physics 712
Chapter 4 Problems



1. [10] A long cylindrical cavity with inner radius *a* and outer radius *c* is filled with a dielectric with dielectric constant ε from radius *a* to radius *b* and with vacuum from radius *b* to radius *c*. If there is charge per unit length λ on the inner conductor, find the electric field everywhere between the cylinders, the potential difference between the inner and outer conductors, and the bound charge density on the surface, if any, of the dielectric at radius *b*.

It makes sense that the electric field and electric displacement would be radial in both regions, so, for example, $\mathbf{D} = D(\rho)\hat{\boldsymbol{\rho}}$. We can then draw a cylinder of radius ρ and length L centered on the axis of the cavity. The charge inside would be λL. If we apply Gauss's Law to this cylinder, there will be no flux through the ends of the cylinder, so we have

$$\lambda L = Q(V) = \int_S \mathbf{D}\cdot\hat{\mathbf{n}}\,da = D(\rho)\int_S da = D(\rho)2\pi\rho L\,,$$

$$\mathbf{D}(\mathbf{x}) = D(\rho)\hat{\boldsymbol{\rho}} = \frac{\lambda\hat{\boldsymbol{\rho}}}{2\pi L}\,.$$

We then use the formula $\mathbf{D} = \varepsilon\mathbf{E}$ to get the electric field, which will be

$$\mathbf{E}(\mathbf{x}) = \begin{cases} \lambda\hat{\boldsymbol{\rho}}/(2\pi\varepsilon\rho) & a<\rho<b\,, \\ \lambda\hat{\boldsymbol{\rho}}/(2\pi\varepsilon_0\rho) & b<\rho<c\,. \end{cases}$$

We should note that, in this case, $\mathbf{D}$ and hence $\mathbf{D}_\perp$ is continuous, and since $\mathbf{E}_\parallel = 0$, it is continuous as well. Also notice that $\mathbf{E}$ is perpendicular at the conductors, as it must be.

Since the electric field is the gradient of the potential, $\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x})$, we can find the potential difference by integrating the electric field, so

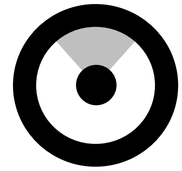
$$\Phi(a)-\Phi(c) = \int_c^a \frac{d}{d\rho}\Phi(\mathbf{x})\,d\rho = \int_a^c E_\rho(\mathbf{x})\,d\rho = \int_a^b \frac{\lambda\,d\rho}{2\pi\varepsilon\rho} + \int_b^c \frac{\lambda\,d\rho}{2\pi\varepsilon_0\rho}$$

$$= \frac{\lambda}{2\pi}\left[\frac{1}{\varepsilon}\ln\rho\Big|_a^b + \frac{1}{\varepsilon}\ln\rho\Big|_b^c\right] = \frac{\lambda}{2\pi}\left[\frac{1}{\varepsilon}\ln\left(\frac{b}{a}\right) + \frac{1}{\varepsilon_0}\ln\left(\frac{c}{b}\right)\right].$$

The bound surface charge density is given by

$$\sigma = \mathbf{P}\cdot\hat{\mathbf{n}} = (\mathbf{D}-\varepsilon_0\mathbf{E})\cdot\hat{\boldsymbol{\rho}} = \left(1-\frac{\varepsilon_0}{\varepsilon}\right)\mathbf{D}\cdot\hat{\boldsymbol{\rho}} = \frac{\lambda}{2\pi b}\left(1-\frac{\varepsilon_0}{\varepsilon}\right).$$

2. [10] A spherical cavity of outer radius *a* and inner radius *b* is partially filled with a dielectric with dielectric constant ε in the region $0 < \theta < \frac{1}{3}\pi$. A charge *Q* is on the inner sphere. Find the electric field everywhere between the spheres, the voltage difference between the inner and outer spheres, and the bound charge density on the surface, if any, of the dielectric at $\theta = \frac{1}{3}\pi$.



We conjecture that in each region, the electric field and electric displacement are radial outwards and depends only on the distance *r* from the center. We need the parallel component of the electric field to be continuous, and since the electric field is parallel to the boundary, we speculate that **E** is the same in both regions, so $\mathbf{E}(\mathbf{x}) = E(r)\hat{\mathbf{r}}$ everywhere between the two conducting shells. The electric displacement will be

$$\mathbf{D}(\mathbf{x}) = \begin{cases} \varepsilon E(r)\hat{\mathbf{r}} & \theta < \frac{1}{3}\pi\,, \\ \varepsilon_0 E(r)\hat{\mathbf{r}} & \theta > \frac{1}{3}\pi\,. \end{cases}$$

We can use Gauss's law on a sphere of radius *r* to find the electric field:

$$Q = \int_S \mathbf{D}(\mathbf{x})\cdot\hat{\mathbf{n}}\,da = r^2\int_0^{2\pi} d\phi \int_0^{\pi} D(r,\theta)\sin\theta d\theta = 2\pi r^2 E(r)\left[\int_0^{\frac{1}{3}\pi}\varepsilon\sin\theta d\theta + \int_{\frac{1}{3}\pi}^{\pi}\varepsilon_0\sin\theta d\theta\right]$$

$$= 2\pi r^2 E(r)\left[-\varepsilon\cos\theta\Big|_0^{\frac{1}{3}\pi} - \varepsilon_0\cos\theta\Big|_{\frac{1}{3}\pi}^{\pi}\right] = 2\pi r^2 E(r)\left[\tfrac{1}{2}\varepsilon + \tfrac{3}{2}\varepsilon_0\right] = \pi(\varepsilon + 3\varepsilon_0) r^2 E(r)\,.$$

We now solve this for *E*(*r*), and we have

$$\mathbf{E}(\mathbf{x}) = \frac{Q\hat{\mathbf{r}}}{\pi(3\varepsilon_0 + \varepsilon) r^2}\,.$$

The potential difference is then

$$\Phi(a) - \Phi(b) = \int_b^a \hat{\mathbf{r}}\cdot\nabla\Phi\,dr = \int_a^b E(r)\,dr = \frac{Q}{\pi(3\varepsilon_0+\varepsilon)}\int_a^b \frac{dr}{r^2} = \frac{Q}{\pi(3\varepsilon_0+\varepsilon)}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$= \frac{Q(b-a)}{\pi ab(3\varepsilon_0+\varepsilon)}\,.$$

The surface charge density on the boundary is $\sigma = \mathbf{P}\cdot\hat{\mathbf{n}} = (\mathbf{D} - \varepsilon_0\mathbf{E})\cdot\hat{\boldsymbol{\theta}} = 0$, since **E** and **D** are both parallel to the boundary.