Physics 712
Chapter 3 Problems

5. [15] A semi-infinite cylinder of radius *a* has potential $\Phi = 0$ on the lateral surface, and $\Phi = V$ on the surface at $z = 0$. Write the potential as an infinite series. Assuming the potential does not diverge as $z \to \infty$, which coefficients must vanish? Find the potential everywhere, and numerically at $\rho = 0$ and $z = a$.

Since the potential vanishes at $\rho = a$, it can be written as a superposition of Bessel functions times $e^{im\phi}$, that is

$$\Phi(\rho,\phi,z) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} A_{mn}(z) J_m\left(\frac{x_{mn}\rho}{a}\right) e^{im\phi}$$

where x_{mn} is the n'th root of J_m. It must satisfy Laplace's equation in the interior, which tells us

$$\begin{aligned}
0 = \nabla^2\Phi &= \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right] A_{mn}(z) J_m\left(\frac{x_{mn}\rho}{a}\right) e^{im\phi} \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im\phi}\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right] A_{mn}(z) J_m\left(\frac{x_{mn}\rho}{a}\right) \\
&= \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{x_{mn}\rho}{a}\right) e^{im\phi}\left[-\frac{x_{mn}^2}{a^2} + \frac{d^2}{dz^2}\right] A_{mn}(z)
\end{aligned}$$

Since $J_m(x_{mn}\rho/a)e^{im\phi}$ forms an independent set of functions on ρ and ϕ, the only way this can be achieved is if the coefficients all vanish, which implies

$$\left[-\frac{x_{mn}^2}{a^2} + \frac{d^2}{dz^2}\right] A_{mn}(z) = 0, \quad \text{or} \quad \frac{d^2}{dz^2} A_{mn}(z) = \frac{x_{mn}^2}{a^2} A_{mn}(z).$$

This equation has two linearly independent solutions, so

$$A_{mn}(z) = \alpha_{mn}\exp\left(\frac{x_{mn}z}{a}\right) + \beta_{mn}\exp\left(-\frac{x_{mn}z}{a}\right).$$

But we only want solutions that do not blow up at infinity, so we demand $\alpha_{mn} = 0$, and then have

$$\Phi(\rho,\phi,z) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} \beta_{mn} J_m\left(\frac{x_{mn}\rho}{a}\right) e^{im\phi}\exp\left(-\frac{x_{mn}z}{a}\right) e^{-x_{mn}z/a}.$$

It remains to find the coefficients β_{mn}. Evaluating this expression at $z = 0$, we must have

$$V = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} \beta_{mn} J_m\left(\frac{x_{mn}\rho}{a}\right) e^{im\phi}.$$

We can use the orthonogonality of the functions $J_m(x_{mn}\rho/a)e^{im\phi}$ to then find:

$$\beta_{mn} = \frac{1}{2\pi}\cdot\frac{2}{a^2 J_{m+1}^2(x_{mn})}\int_0^{2\pi} e^{-im\phi}\,d\phi\int_0^a VJ_m\left(\frac{x_{mn}\rho}{a}\right)\rho\,d\rho = \frac{2V\delta_{m0}}{a^2 J_1^2(x_{0n})}\int_0^a J_0\left(\frac{x_{0n}\rho}{a}\right)\rho\,d\rho$$

$$= \frac{2V\delta_{m0}}{x_{0n}^2 J_1^2(x_{0n})}\int_0^{x_{0n}} J_0(y)\,y\,dy\,,$$

where in the last step we made the substitution $\rho = ay/x_{0n}$. You can get Maple to do the integrals numerically, or if you want to be cleverer, you can use the recursion relations

$$J_{m\pm1}(x) = \frac{m}{x}J_m(x) \mp \frac{d}{dx}J_m(x)$$

to rewrite the final integral as

$$\int_0^{x_{0n}} J_0(y)\,y\,dy = \int_0^{x_{0n}}\left[\frac{1}{y}J_1(y) + \frac{d}{dy}J_1(y)\right]y\,dy = \int_0^{x_{0n}}\frac{d}{dy}\left[yJ_1(y)\right]dy = yJ_1(y)\Big|_0^{x_{0n}} = x_{0n}J_1(x_{0n})\,.$$

We therefore find $\beta_{0n} = 2V/\left[x_{0n}J_1(x_{0n})\right]$. The first several terms can be found in the table at right. The potential is

n	β_{0n}/V
1	1.601974697
2	-1.064799259
3	0.851399193
4	-0.729645240
5	0.648523614
6	-0.589542829
7	0.544180196
8	-0.507893631
9	0.478012498

$$\Phi(\rho,\phi,z) = \sum_{n=1}^{\infty}\frac{2V}{x_{0n}J_1(x_{0n})}J_0\left(\frac{x_{0n}\rho}{a}\right)\exp\left(-\frac{x_{0n}z}{a}\right).$$

Substituting in $z = a$ and using the fact that $J_0(0) = 1$, we have

$$\Phi(0,\phi,a) = \sum_{n=1}^{\infty}\frac{2Ve^{-x_{0n}}}{x_{0n}J_1(x_{0n})} = 0.1405064065V\,.$$

We let Maple do the sum for us. The numerical value was found adding seven terms.

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> add(2/exp(x)/x/BesselJ(1,x),x=evalf(BesselJZeros(0,1..7)));
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