Physics 712
Chapter 1 Problems

5. [20] Consider the region *z* > 0, with boundary condition $\partial\Phi/\partial n = 0$ on the boundary.

(a) Show that the Green function for this region and boundary condition is given by

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} + \frac{1}{|\mathbf{x}_R-\mathbf{x}'|}, \quad \text{where} \quad \mathbf{x}_R = (x, y, -z)$$

We have to check that the Laplacian is appropriate, and that it satisfies Neumann boundary conditions on the boundary. We have:

$$\nabla'^2 G(\mathbf{x},\mathbf{x}') = \nabla'^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|} + \nabla'^2 \frac{1}{|\mathbf{x}_R-\mathbf{x}'|} = -4\pi\delta^3(\mathbf{x}-\mathbf{x}') - 4\pi\delta^3(\mathbf{x}_R-\mathbf{x}').$$

The first term is the one we want. The second one is non-vanishing only when $\mathbf{x}' = \mathbf{x}_R$, but since both **x** and **x**′ are supposed to be in the allowed region, z' will be positive while z_R is negative, and hence this delta function vanishes everywhere in the allowed region. So that part worked out.

We also have to check the boundary condition, which requires us to check the normal derivative at $z' = 0$. Since the normal is supposed to point *out* of the region the normal derivative will be $-\partial/\partial z'$. Writing everything out explicitly in Cartesian coordinates, we have

$$\begin{aligned}
\frac{\partial}{\partial n'} G(\mathbf{x},\mathbf{x}')\Big|_{z'=0} &= -\frac{\partial}{\partial z'}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right]_{z'=0} \\
&= \left[\frac{z'-z}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}} + \frac{z'+z}{\left[(x-x')^2+(y-y')^2+(z+z')^2\right]^{3/2}}\right]_{z'=0} \\
&= 0.
\end{aligned}$$

That was painful, but it's done.

(b) Find the electric potential everywhere if there is a line of charge along part of the *z*-axis, so that

$$\rho(\mathbf{x}) = \begin{cases} \lambda\delta(x)\delta(y) & \text{for } z < a, \\ 0 & \text{for } z > a. \end{cases}$$

Since we have zero boundary conditions, the potential is just given by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int G(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')d^3\mathbf{x}'$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\delta(x')dx'\int_{-\infty}^{\infty}\delta(y')dy'\int_0^a dz'\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\int_0^a dz'\left[\frac{1}{\sqrt{x^2+y^2+(z-z')^2}} + \frac{1}{\sqrt{x^2+y^2+(z+z')^2}}\right]$$

On the last term in the integral, you can change variables $z' \to -z'$, which makes the two terms in the integral identical, but the range for z' will be [–a,0] now, so they can be combined to yield

$$\Phi(\mathbf{x}) = \frac{\lambda}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{dz'}{\sqrt{x^2+y^2+(z-z')^2}} = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{\sqrt{x^2+y^2+(z-a)^2}+a-z}{\sqrt{x^2+y^2+(z+a)^2}-a-z}\right)$$

The final integral isn’t too hard to do by hand, but I let Maple do it for me.