Physics 712
Chapter 1 Problems

3. [10] Is it possible to design a trap for a particle? Suppose we wish to trap a particle with positive charge q at the origin, but such that there is no other charge very close, so that $\rho = 0$ in a small neighborhood of the origin.

(a) What condition(s) can you place on the potential $\Phi(\mathbf{x})$ or its derivative so that it is at a local minimum of energy in the *x*-direction. Repeat for the *y* and *z*-direction.

Since the charge is positive, the potential energy $q\Phi(\mathbf{x})$ will have its minimum at the same places $\Phi(\mathbf{x})$ does. To have a minimum in the *x*-direction, the first *x*-derivative must vanish and the second derivative must be positive. Of course, the same rules apply in the other two directions, so we have

$$\frac{\partial}{\partial x}\Phi(\mathbf{x})\Big|_0 = \frac{\partial}{\partial y}\Phi(\mathbf{x})\Big|_0 = \frac{\partial}{\partial z}\Phi(\mathbf{x})\Big|_0 = 0, \quad \frac{\partial^2}{\partial x^2}\Phi(\mathbf{x})\Big|_0 > 0, \quad \frac{\partial^2}{\partial y^2}\Phi(\mathbf{x})\Big|_0 > 0, \quad \frac{\partial^2}{\partial z^2}\Phi(\mathbf{x})\Big|_0 > 0.$$

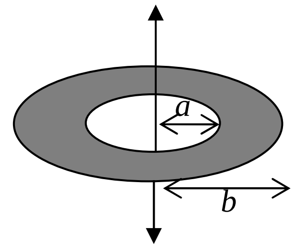
(b) Given that $\rho = 0$ at the origin, prove or disprove that you can have a local minimum of the potential there.

Since $\nabla^2\Phi = -\rho/\varepsilon_0$, and $\rho = 0$, we must have $\nabla^2\Phi = 0$. But if we add the three inequalities from part (a), we see that

$$\nabla^2\,\Phi(\mathbf{x})\Big|_0 = \frac{\partial^2}{\partial x^2}\Phi(\mathbf{x})\Big|_0 + \frac{\partial^2}{\partial y^2}\Phi(\mathbf{x})\Big|_0 + \frac{\partial^2}{\partial z^2}\Phi(\mathbf{x})\Big|_0 > 0.$$

Since this is a contradiction, we conclude that it is impossible to make an electrostatic trap.

4. **[10] Consider an annulus (hollow circle) of surface charge density σ with inner radius *a* and outer radius *b* centered on the origin in the *xy*-plane. Find the potential and electric field everywhere on the *z*-axis. If a charge *q*, initially at rest, were released from the origin, what would be its speed *v* when it reaches infinity?**



The distance between a point on the annulus that is a distance ρ from the origin and a point at *z* on the *z*-axis is given by $|\mathbf{x}-\mathbf{x}'| = \sqrt{\rho^2+z^2}$. The potential is given by

$$\Phi(z\hat{\mathbf{z}}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x} = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi \int_a^b \rho d\rho \frac{\sigma}{\sqrt{\rho^2+z^2}} = \frac{\sigma}{2\varepsilon_0}\sqrt{\rho^2+z^2}\Big|_a^b$$

$$= \frac{\sigma}{2\varepsilon_0}\left(\sqrt{b^2+z^2}-\sqrt{a^2+z^2}\right).$$

It is pretty easy to see from the symmetry of the problem that the electric field will point in the *z*-direction, so we have

$$\mathbf{E}(z\hat{\mathbf{z}}) = -\nabla\Phi = -\hat{\mathbf{z}}\frac{\sigma}{2\varepsilon_0}\frac{\partial}{\partial z}\left(\sqrt{b^2+z^2}-\sqrt{a^2+z^2}\right) = \frac{\sigma z\hat{\mathbf{z}}}{2\varepsilon_0}\left(\frac{1}{\sqrt{a^2+z^2}}-\frac{1}{\sqrt{b^2+z^2}}\right).$$

If a particle at rest with positive charge *q* is placed at the origin, it will initially have potential energy $W = q\Phi(0) = q\sigma(b-a)/(2\varepsilon_0)$. Though there is no force on the particle, it is at a point of unstable equilibrium, and will quickly slip away and accelerate to infinity. At infinity, all of the potential energy has been converted to kinetic energy, so that the final kinetic energy will have

$$\frac{mv^2}{2} = \frac{q\sigma(b-a)}{2\varepsilon_0},$$

$$v = \sqrt{\frac{q\sigma(b-a)}{m\varepsilon_0}}.$$