Physics 310/610 – Cosmology

Solution Set W

1. **[15] This question concerns the relative strength of electric and gravitational forces.**
   (a) Write a formula for the gravitational force between two electrons. Find the ratio of the gravitational force to the Coulomb force, $F = ke^2/r^2$, where *k* is Coulomb's constant and *e* is the electron charge (which you can go look up). Show that the ratio is constant and evaluate it.

The gravitational force between two particles is $F_g = Gm^2/r^2$, and the ratio of these forces is

$$\frac{F_g}{F_E} = \frac{Gm^2/r^2}{ke^2/r^2} = \frac{Gm^2}{ke^2} = \frac{\left(6.674\times10^{-11}\ \text{m}^3/\text{kg/s}^2\right)\left(9.1094\times10^{-31}\ \text{kg}\right)^2}{\left(8.9876\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(1.602\times10^{-19}\ \text{C}\right)^2} = 2.40\times10^{-43}$$

As we can see, at conventional energies (such as the electron mass), gravity is far weaker than other forces such as the electric force.

(b) For very energetic particles, the mass *m* for the electron is replaced by E/c^2. For what energy *E* in GeV will the gravitational force between a pair of electrons be as strong as the electric force?

The electric force does not change, because it depends only on the electric charge. The two forces will match when

$$1 = \frac{F_g}{F_E} = \frac{Gm^2/r^2}{ke^2/r^2} = \frac{G\left(E/c^2\right)^2}{ke^2} = \frac{GE^2}{ke^2c^4},$$

$$E^2 = \frac{ke^2c^4}{G} = \frac{\left(8.9876\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(1.602\times10^{-19}\ \text{C}\right)^2\left(2.998\times10^{8}\ \text{m/s}\right)^4}{6.674\times10^{-11}\ \text{m}^3/\text{kg/s}^2} = 2.792\times10^{16}\ \text{J}^2,$$

$$E = \sqrt{2.792\times10^{16}\ \text{J}^2} = 1.671\times10^{8}\ \text{J} = \frac{1.671\times10^{8}\ \text{J}}{1.602\times10^{-19}\ \text{eV/J}} = 1.043\times10^{27}\ \text{eV} = 1.043\times10^{18}\ \text{GeV}.$$

(c) A typical energy of a particle at temperature *T* is given by $3k_BT$. What is k_BT when gravity is the same strength as the other forces? How old is the universe at this time? Assume the universe is radiation dominated at this time, with $g_{\text{eff}} = 200$ for definiteness.

Obviously, $k_BT = \frac{1}{3}E = 3.48\times10^{17}\ \text{GeV}$. We then simply substitute this into the formula for the age of the universe.

$$t = \frac{2.42\text{ s}}{\sqrt{g_{\text{eff}}}}\left(\frac{\text{MeV}}{k_B T}\right)^2 = \frac{2.42\text{ s}}{\sqrt{200}}\left(\frac{\text{MeV}}{3.48\times10^{20}\text{ MeV}}\right)^2 = 1.41\times10^{-42}\text{ s}$$

2. **[15] We have recently been discussing things like the *Planck mass*, the *Planck time*, and the *Planck length*. What are all these quantities? Any quantum theory of gravity must involve the speed of light *c*, the reduced Planck constant $\hbar$ and the gravitational constant *G*. Start by looking up the units of each of these quantities (unless you know them).**
   (a) **Using only dimensional analysis, find expressions for the Planck time t_P, the Planck distance l_P and the Planck mass m_P, where in each case the formula will be of the form $x_P = G^\alpha \hbar^\beta c^\gamma$, where α, β, and γ are simple rational numbers.**

   The three constants are

$$G = 6.674\times10^{-11}\text{ m}^3/\text{kg/s}^2,\quad \hbar = 1.0546\times10^{-34}\text{ kg}\cdot\text{m}^2/\text{s},\quad c = 2.998\times10^8\text{ m/s}.$$

We first note that if we want kg to *not* appear in an expression, we had better have *G* and $\hbar$ to the same power, so that kg will cancel. Noting that $G\hbar \sim \text{m}^5/\text{s}^3$, we see that we can eliminate seconds by dividing by c^3, so that $G\hbar/c^3 \sim \text{m}^2$. The square root of this must be the Planck length, $l_P = \sqrt{G\hbar/c^3}$. Dividing by one more factor of *c* turns meters into seconds, so we have $t_P = \sqrt{G\hbar/c^3}\big/c = \sqrt{G\hbar/c^5}$. It is then pretty easy to get units that look like kilograms; for example, $\hbar t_P/l_P^2 \sim \text{kg}$, we must have $m_P = \hbar t_P/l_P^2$. Putting it all together, we have

$$l_P = \sqrt{\frac{G\hbar}{c^3}},\quad t_P = \sqrt{\frac{G\hbar}{c^5}},\quad m_P = \frac{\hbar t_P}{l_P^2} = \hbar\sqrt{\frac{G\hbar}{c^5}}\frac{c^3}{G\hbar} = \sqrt{\frac{\hbar c}{G}}.$$

   (b) **Evaluate each of the quantities in part (a) in standard SI units.**

   We simply plug into the previous formula

$$l_P = \sqrt{\frac{\left(6.674\times10^{-11}\text{ m}^3/\text{kg/s}^2\right)\left(1.0546\times10^{-34}\text{ kg}\cdot\text{m}^2/\text{s}\right)}{\left(2.998\times10^8\text{ m/s}\right)^3}} = 1.616\times10^{-35}\text{ m},$$

$$t_P = \sqrt{\frac{\left(6.674\times10^{-11}\text{ m}^3/\text{kg/s}^2\right)\left(1.0546\times10^{-34}\text{ kg}\cdot\text{m}^2/\text{s}\right)}{\left(2.998\times10^8\text{ m/s}\right)^5}} = 5.39\times10^{-44}\text{ s},$$

$$m_P = \sqrt{\frac{\left(1.0546\times10^{-34}\text{ kg}\cdot\text{m}^2/\text{s}\right)\left(2.998\times10^8\text{ m/s}\right)}{6.674\times10^{-11}\text{ m}^3/\text{kg/s}^2}} = 2.176\times10^{-8}\text{ kg}.$$

(c) Using simple combinations from parts (a) and (b), find the Planck energy E_P in both J and GeV, and find the Planck mass density ρ_P in kg/m³. Naively, the mass density of empty space should be about ρ_P. What is the ratio of ρ_P to the actual mass density of empty space, $\rho_\Lambda = 5.65\times10^{-27}\ \text{kg/m}^3$?

The Planck energy is just given by $E_P = m_P c^2$, so we have

$$E_P = m_P c^2 = \left(2.176\times10^{-8}\ \text{kg}\right)\left(2.998\times10^{8}\ \text{m/s}\right)^2 = 1.957\times10^{9}\ \text{J} = 1.957\ \text{GJ}$$

$$= \frac{1.956\times10^{9}\ \text{J}}{1.602\times10^{-19}\ \text{eV/J}} = 1.221\times10^{28}\ \text{eV} = 1.221\times10^{19}\ \text{GeV}\,.$$

The Planck mass density would be

$$\rho_P = \frac{m_P}{l_P^3} = \frac{2.176\times10^{-8}\ \text{kg}}{\left(1.616\times10^{-35}\ \text{m}\right)^3} = 5.156\times10^{96}\ \text{kg/m}^3\,,$$

$$\frac{\rho_P}{\rho_\Lambda} = \frac{5.156\times10^{96}\ \text{kg/m}^3}{5.65\times10^{-27}\ \text{kg/m}^3} = 9.13\times10^{122}\,.$$

As you can see, these numbers are rather different.

(d) Suppose it actually takes an energy E_P to create a universe. Given that the cost of electricity in the United States in 2023 is about \$0.169 per kilowatt hour, how much would it cost you to make a universe?

$$\text{cost} = (\text{unit})\cdot(\text{cost/unit}) = \left(1.957\times10^{9}\,\text{J}\right)\cdot\frac{\$0.169}{\text{kW}\cdot\text{hr}}\cdot\frac{\text{hr}}{3600\ \text{s}}\cdot\frac{1\ \text{kW}}{1000\ \text{W}}\cdot\frac{\text{W}}{\text{J/s}} = \$91.87\,.$$

Something to think about when trying to come up with that special Christmas gift. How about giving someone a whole universe of their own?