Physics 310/610 – Cosmology

Solution Set V

1. **[15] In class we found that the annihilation cross-section required to get the correct density of dark matter today is $\sigma \approx 4.5\times10^{-40}\ \text{m}^2$.**
   (a) [5] Assume the cross section takes a typical electromagnetic cross-section value, $\sigma = \alpha^2\hbar^2/(m^2c^2)$, where $\alpha = \frac{1}{137}$ is the fine structure constant. What would the relevant mass *m* in GeV/c^2 be?

   Let's find the mass in terms of the cross-section. We have $\sqrt{\sigma} = \alpha\hbar/(mc)$

$$mc^2 = \frac{\alpha\hbar c}{\sqrt{\sigma}} = \frac{(197.33\ \text{eV}\cdot\text{nm})(10^{-9}\ \text{m/nm})}{137\sqrt{4.5\times10^{-40}\ \text{m}^2}} = 6.79\times10^{10}\ \text{eV} = 67.9\ \text{GeV}.$$

If this were truly the mass, and it had charge e¸ we would have easily already discovered it.

   (b) [5] Suppose the particle annihilates via weak interactions, with cross-sections of order $\sigma = G_F^2 E_1 E_2/(\hbar c)^4$, where $G_F/(\hbar c)^3 = 1.166\times10^{-5}\ \text{GeV}^{-2}$. What mass would you need? Keep in mind that the particles are non-relativistic when they collide.

   Since the particles are non-relativistic, their energy will be $E_1 = E_2 = mc^2$. We therefore have

$$\sigma = \frac{G_F^2 (mc^2)^2}{(\hbar c)^4}$$

$$mc^2 = \sqrt{\frac{(\hbar c)^4 \sigma}{G_F^2}} = \frac{(\hbar c)^2\sqrt{\sigma}}{G_F} = \frac{(\hbar c)^3\sqrt{\sigma}}{G_F(\hbar c)} = \frac{\sqrt{4.5\times10^{-40}\ \text{m}^2}\,(10^9\ \text{eV/GeV})^2(10^9\ \text{nm/m})}{(1.166\times10^{-5}\ \text{GeV}^{-2})(197.33\ \text{eV}\cdot\text{nm})},$$

$$mc^2 = 9.22\times10^9\ \text{eV} = 9.22\ \text{GeV}.$$

Even though these particles are only made weakly, we should have produced lots of them in colliders by now. Though it's not obvious how they would be detected, I doubt they could escape detection.

(c) [5] There is an approximate maximum cross-section for annihilation $\sigma_{\max} = 4\pi\hbar^2/p^2$, where p is the momentum of the particles. Assume the kinetic energy of a typical dark-matter particle at freezeout is $\frac{3}{2}k_BT$, and that $k_BT = \frac{1}{30}mc^2$. What is the maximum mass that the dark matter particles could have and keep their cross section below $\sigma_{\max}$?

The kinetic energy is given by $E_{kin} = p^2/(2m)$, so $p^2 = 2mE_{kin} = 3mk_BT = \frac{1}{10}m^2c^2$. We therefore have

$$\sigma_{\max} = \frac{4\pi\hbar^2}{p^2} = \frac{4\pi\hbar^2 10}{m^2c^2},$$

$$m^2c^2 = \frac{40\pi\hbar^2}{\sigma_{\max}},$$

$$mc^2 = \sqrt{\frac{40\pi}{\sigma_{\max}}}\hbar c = \frac{\sqrt{40\pi}\left(197.33\times10^{-9}\text{ eV}\cdot\text{m}\right)}{\sqrt{4.5\times10^{-40}\text{ m}^2}} = 1.043\times10^{14}\text{ eV} = 104.3\text{ TeV}.$$

2. [15] The universe contains a lot of energy, but it grew a lot during inflation. How much energy was there before?

(a) [4] The energy in radiation is given by

$$u = g_{eff}\frac{\pi^2\left(k_BT\right)^4}{30\left(\hbar c\right)^3}$$

Calculate this energy density in J/m³ both today (g_{eff} = 3.36) and at the end of inflation (k_BT = 10¹⁶ GeV, and we'll guess $g_{eff} = 200$).

The energy density now (treating the neutrinos as radiation, even though they really aren't today) is

$$u_0 = \frac{3.36\pi^2\left[\left(1.3806\times10^{-23}\text{ J/K}\right)\left(2.725\text{ K}\right)\right]^4}{30\left[\left(1.055\times10^{-34}\text{ J}\cdot\text{s}\right)\left(2.998\times10^8\text{ m/s}\right)\right]^3} = 7.009\times10^{-14}\text{ J/m}^3,$$

$$u_{\text{GUT}} = \frac{200\pi^2\left[\left(10^{25}\text{ eV}\right)\left(1.602\times10^{-19}\text{ J/eV}\right)\right]^4}{30\left[\left(1.055\times10^{-34}\text{ J}\cdot\text{s}\right)\left(2.998\times10^8\text{ m/s}\right)\right]^3} = 1.37\times10^{103}\text{ J/m}^3.$$

(b) [4] The current size of the visible universe is a sphere of radius about 13.5 Gpc. Convert this to meters. To a very good approximation, the scale factor is related to the size of the universe by the approximate relation $a \propto T^{-1}g_{eff}^{-1/3}$. What was the size of the region that become the *current* visible universe at the end of inflation?

The current size of the universe is about

$$a_0 = \left(13.5\times10^9 \text{ pc}\right)\left(3.0857\times10^{16} \text{ m/pc}\right) = 4.166\times10^{26} \text{ m}.$$

The value at the GUT scale would be

$$a_{\text{GUT}} = a_0\left(\frac{T_0}{T_{\text{GUT}}}\right)\left(\frac{g_0}{g_{\text{GUT}}}\right)^{1/3} = a_0\left(\frac{k_B T_0}{k_B T_{\text{GUT}}}\right)\left(\frac{g_0}{g_{\text{GUT}}}\right)^{1/3}$$

$$= \left(4.166\times10^{26} \text{ m}\right)\frac{\left(1.3806\times10^{-23} \text{ J/K}\right)\left(2.725 \text{ K}\right)}{\left(10^{25} \text{ eV}\right)\left(1.602\times10^{-19} \text{ J/eV}\right)}\left(\frac{3.36}{200}\right)^{1/3} = 0.00251 \text{ m}.$$

The entire visible universe came from a region that (at the time) was 2.5 mm in radius.

(c) [4] Find the total radiation energy of the universe now and at the end of inflation (in J).

In each case, we simply multiply the energy density by the volume of the universe, $\frac{4}{3}\pi a^3$. We have

$$E_0 = \tfrac{4}{3}\pi a_0^3\rho_0 = \tfrac{4}{3}\pi\left(4.166\times10^{26} \text{ m}\right)^3\left(7.009\times10^{-14} \text{ J/m}^3\right) = 2.12\times10^{67} \text{ J},$$

$$E_{\text{GUT}} = \tfrac{4}{3}\pi a_{\text{GUT}}^3\rho_{\text{GUT}} = \tfrac{4}{3}\pi\left(0.00251 \text{ m}\right)^3\left(1.37\times10^{103} \text{ J/m}^3\right) = 9.07\times10^{95} \text{ J}.$$

The universe was small but mighty at the time.

(d) [3] According to inflation, the universe grew in size by at least a factor of 10^{28}. Assuming the temperature at the start of inflation was the same as at the end, calculate the energy at the start of inflation. Then work out the equivalent mass, using $E = mc^2$. You should find that the mass required is remarkably small.

The density will be the same; the only difference is that the universe will be 10^{28} times smaller in all three dimensions. Therefore the energy pre-inflation will be

$$E_{\text{pre-inf}} = \left(10^{28}\right)^{-3} E_{\text{GUT}} = 10^{-84}\left(9.07\times10^{95} \text{ J}\right) = 9.07\times10^{11} \text{ J} = 907 \text{ GJ},$$

$$m_{\text{pre-inf}} = \frac{E_{\text{pre-inf}}}{c^2} = \frac{9.07\times10^{11} \text{ J}}{\left(2.998\times10^{8} \text{ m/s}\right)^2} = 1.01\times10^{-5} \text{ kg} = 0.0101 \text{ g}.$$

The mass is trivial, and the energy, though large, is not inconceivably large.

Graduate Problems: There are no problems for PHY 610 on this homework