Physics 310/610 – Cosmology

Solution Set O

1. **[15] In class, we found that in the future, the size of the universe will grow exponentially, $a \propto \exp(H_\Lambda t)$.**

   (a) Using our best estimates of H_0 and Ω_Λ, find H_Λ in Gyr^{-1}. A good estimate of the distance to the edge of the visible universe at that time would be $d_{\max} = c/H_\Lambda$. Find $d_{\max}$ in Gpc.

   In the notes, we had $a \propto \exp\left(H_0 t\sqrt{\Omega_\Lambda}\right)$, so clearly $H_\Lambda = H_0\sqrt{\Omega_\Lambda}$, so

$$
\begin{aligned}
H_\Lambda &= H_0\sqrt{\Omega_\Lambda} = (67.7\text{ km/s/Mpc})\sqrt{0.6889} = 56.2\,\text{km/s/Mpc} \\
&= (56.2\,\text{km/s/Mpc})\frac{\left(3.156\times10^{7}\text{ s/yr}\right)\left(10^{3}\,\text{m/km}\right)}{\left(3.086\times10^{16}\text{ m/pc}\right)\left(10^{6}\text{ pc/Mpc}\right)} \\
&= 5.75\times10^{-11}\text{ yr}^{-1} = 0.0575\text{ Gyr}^{-1}\,.
\end{aligned}
$$

The distance $d_{\max}$ can then be calculated as

$$
d_{\max} = \frac{c}{H_\Lambda} = \frac{2.998\times10^{5}\text{ km/s}}{56.2\text{ km/s/Mpc}} = 5330\text{ Mpc} = 5.33\text{ Gpc}.
$$

   (b) At present, the nearest galactic cluster to the local group is about at a distance of 3.3 Mpc. Assuming it participates in the general expansion of the universe, how far in the future will it be until it reaches the distance $d_{\max}$.

   We will use time coordinates such that $t = 0$ corresponds to now. Therefore, if d_0 is the distance to an object now, then $d(t) = d_0\exp(H_\Lambda t)$. Solving for t, we have

$$
t = H_\Lambda^{-1}\log(d/d_0).
$$

For this nearby galaxy, we have

$$
t = H_\Lambda^{-1}\ln\left(\frac{5.33\text{ Gpc}}{3.3\text{ Mpc}}\cdot\frac{10^{3}\text{ Mpc}}{\text{Gpc}}\right) = \frac{\ln(1615)}{0.0575\text{ Gyr}^{-1}} = 128\text{ Gyr}.
$$

At this point, seeing other galaxies is going to get very difficult.

   (c) We know about the big bang largely because of the cosmic microwave background radiation. Find the peak wavelength for the $\lambda_{\max}$ for the current cosmic microwave background radiation. This radiation is theoretically undetectable when $\lambda_{\max}$ exceeds $d_{\max}$, due to the expansion of the Universe. How long in the future will this occur?

We simply start with Wien's formula, $\lambda_{\max} T = 2.898\times10^{-3}\ \mathrm{m\cdot K}$, to find

$$\lambda_{\max} = \frac{2.898\times10^{-3}\ \mathrm{m\cdot K}}{T} = \frac{2.898\times10^{-3}\ \mathrm{m\cdot K}}{2.725\ \mathrm{K}} = 1.063\times10^{-3}\ \mathrm{m} = 1.063\ \mathrm{mm}.$$

We then work out the time as before:

$$t = H_\Lambda^{-1} \ln\left(\frac{5.33\ \mathrm{Gpc}}{1.063\times10^{-3}\ \mathrm{m}} \cdot \frac{3.0857\times10^{25}\ \mathrm{m}}{\mathrm{Gpc}}\right) = \frac{\ln\left(1.547\times10^{29}\right)}{0.0575\ \mathrm{Gyr}^{-1}} = 1170\ \mathrm{Gyr}.$$

It is pretty clear this will be invisible about a trillion years from now.

Event	z	T (K)	$k_B T$ (eV)	Age
Reionization	**10.5**	31.3	0.00270	441 Myr
Room Temp	109	**300.**	0.0259	15.0 Myr
Recombination	1089	2970	**0.256**	481 kyr

2. [15] Estimate the age of the universe (in convenient multiples of the year), the red shift z, the temperature T in K, and the characteristic energy $k_B T$ for each of the following events:

(a) Reionization of the universe at z = 10.5.

$$T = T_0(1+z) = (2.725\ \mathrm{K})(11.5) = 31.3\ \mathrm{K},$$
$$k_B T = \left(8.6173\times10^{-5}\ \mathrm{eV/K}\right)(31.3\,\mathrm{K}) = 0.00270\ \mathrm{eV},$$
$$t = (17.3\ \mathrm{Gyr})(1+z)^{-3/2} = (17.3\ \mathrm{Gyr})(11.5)^{-3/2} = 0.441\ \mathrm{Gyr} = 441\ \mathrm{Myr}.$$

(b) Universe is at room temperature T = 300 K.

$$1+z = T/T_0 = 300/2.725 = 110,$$
$$k_B T = \left(8.6173\times10^{-5}\ \mathrm{eV/K}\right)(300\,\mathrm{K}) = 0.0259\ \mathrm{eV},$$
$$t = (17.3\ \mathrm{Gyr})(1+z)^{-3/2} = (17.3\ \mathrm{Gyr})(110)^{-3/2} = 0.0150\ \mathrm{Gyr} = 15.0\ \mathrm{Myr}.$$

(c) Recombination $k_B T = 0.256\ \mathrm{eV}$.

$$T = k_B T/k_B = 0.256\ \mathrm{eV}\Big/\left(8.6173\times10^{-5}\ \mathrm{eV/K}\right) = 2970\ \mathrm{K}$$
$$1+z = T/T_0 = 2970/2.725 = 1090,$$
$$k_B T = \left(8.6173\times10^{-5}\ \mathrm{eV/K}\right)(300\,\mathrm{K}) = 0.0259\ \mathrm{eV},$$
$$t = (17.3\ \mathrm{Gyr})(1+z)^{-3/2} = (17.3\ \mathrm{Gyr})(1090)^{-3/2} = 481\ \mathrm{kyr}.$$

The time in this case is actually a bit off. The universe is younger than this because we are moving dangerously close to the era of radiation dominance, and hence have significantly overestimated the age of the universe.

3. [10] The number density of photons in a thermal distribution is given by

$$n_\gamma = \frac{2\zeta(3)}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \quad \text{where} \quad \zeta(3) = \sum_{n=1}^{\infty}\frac{1}{n^3} \approx 1.202$$

(a) Find a general formula for the average energy of a photon, given by $\bar{E} = u/n$. Hint: your instructor uses the approximation $\bar{E} = 3k_B T$.

Looking back at our formulas, we have

$$u = \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3}$$

Taking the ratio, we have

$$\bar{E} = \frac{u}{n} = \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3}\frac{\pi^2}{2\zeta(3)}\left(\frac{\hbar c}{k_B T}\right)^3 = \frac{\pi^4}{30\zeta(3)}k_B T = 2.70 k_B T$$

Obviously, your professor is a lazy bum who is unwilling to get the number more accurately, since the number he uses is about ten percent off.

(b) Find the current density of background photons in the universe, and the ratio of photons to baryons, n_B/n_γ.

$$n_\gamma = \frac{2\zeta(3)}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 = \frac{2\zeta(3)}{\pi^2}\left[\frac{(8.6173\times10^{-5}\text{ eV/K})(2.725\text{ K})}{1.9733\times10^{-7}\text{ eV}\cdot\text{m}}\right]^3 = 4.10\times10^{8}\text{ m}^{-3},$$

$$\frac{n_B}{n_\gamma} = \frac{0.251}{4.10\times10^{8}} = 6.12\times10^{-10}.$$

Note that this ratio, which is called η, is constant in the expanding universe.

Graduate Problem: Only do this problem if you are in PHY 610

4. [15] The 4d metric (assuming the universe is flat) is given by

$$ds^2 = -c^2 dt^2 + a^2(t)\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\right)$$

where in the future, $a(t) \approx a_0 \exp(H_\Lambda t)$, where a_0 is the size of the universe now, and t is the time starting from now.

(a) Suppose we have an incoming photon moving directly towards us (photons always have ds = 0). Find an equation for dr/dt.

Since it is coming radially at us, we have $d\theta = d\phi = 0$. Setting also ds = 0, we have $c^2 dt^2 = a^2(t)\, dr^2$. Taking the square root, we have

$$\frac{dr}{dt} = -\frac{c}{a(t)} = -\frac{c}{a_0}\exp(-H_\Lambda t)$$

We were careful to choose the negative square root since the object is coming towards us.

(b) Solve the equation from part (a) so you can get $r(t)$ for an incoming photon.

Integrating this equation is trivial, though we have to remember the constant of integration:

$$r = \frac{c}{a_0 H_\Lambda}\exp(-H_\Lambda t) + k.$$

(c) Show that at *any* time in the future, there is a distance d_{max} such that a photon leaving from d_{max} at time t will never reach us. The distance to an object at time t is given by $ra(t)$. You should find that d_{max} is independent of time.

An object will reach us only if $r = 0$. If $k \geq 0$, then it is obvious that the equation above can never vanish, whereas if $k < 0$, then it is easy to find a time when $r = 0$. It follows that the maximum r you can start with would be when $k = 0$, so

$$r_{max} = \frac{c}{a_0 H_\Lambda}\exp(-H_\Lambda t).$$

At this time, the distance will be

$$d_{max} = r_{max} a(t) = \frac{c}{a_0 H_\Lambda}\exp(-H_\Lambda t)\, a_0 \exp(H_\Lambda t) = \frac{c}{H_\Lambda}.$$

This is the same value we used as a supposed approximation in problem 1.