Physics 310/610 – Cosmology

Solution Set L

1. [20] A set of Type Ia supernovae has their peak apparent magnitude *m* and their red shift *z* measured. The results are collected in the table at right. Assume, for purposes of this problem, that Type Ia supernovae have a peak absolute magnitude of $M_{\max} = -19.3$.

$m_{\max}$	z	d (Mpc)	v (km/s)
7.3	–.0003	2.09	– 90
11.6	0.0038	15.1	1140
14.3	0.0121	52.5	3630
15.7	0.0247	100	7410
16.1	0.0295	120	8840
16.6	0.0363	151	10880
16.7	0.0383	158	11480
17.2	0.0476	200	14270

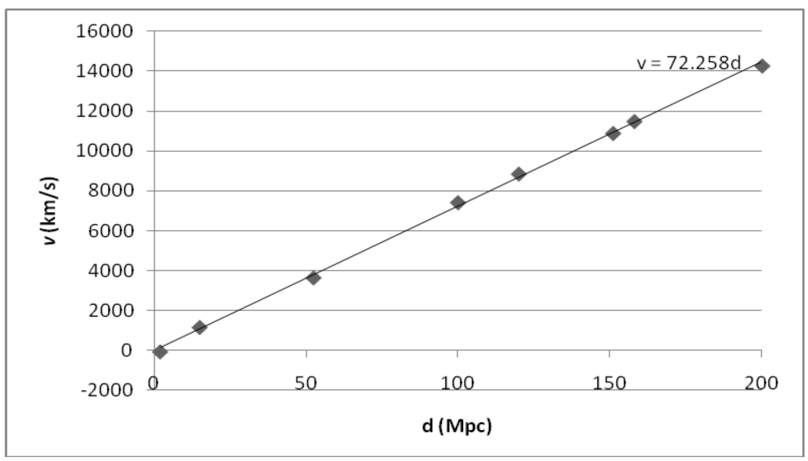
(a) [10] For each supernova, work out the distance *d* in Mpc and the radial velocity *v* in km/s. For the velocity, I recommend using the non-relativistic approximation to save time.

All of the supernovae have small red shift, so we can use the non-relativistic approximation $z = v/c$, which gives us velocities of $v = zc = 299{,}800z$ km/s. The resulting velocities are filled into the table. The distances can be found with

$$d = 10^{1+\frac{m-M}{5}}\ \text{pc} = 10^{1+\frac{m+19.3}{5}}\ \text{pc} = 10^{-5+\frac{m+19.3}{5}}\ \text{Mpc} = 10^{(m-5.7)/5}\ \text{Mpc}$$

The resulting distances and velocities have been filled into the table.

(b) [6] Plot the velocity versus the distance for this set of points. Why are the points not exactly on a straight line?



Excel is happy to plot the data for us, and even provides a trendline. It is a little non-trivial to figure out if we want to force the straight line fit to go through zero or not. Obviously, we aren't moving compared to ourselves (which argues for forcing it to be zero), but on the other hand, we might have a peculiar velocity (which argues for allowing an offset).

The points are not exactly on a straight line because of peculiar velocities; that is, galaxies have velocities that differ slightly from the overall Hubble flow. Indeed, one of the stars is so close to us that it is actually moving towards us.

(c) [4] Estimate the value of Hubble's constant H_0, in km/s/Mpc.

As you can see from the graph (or Excel's fit), Hubble's constant is about 72.26 km/s/Mpc. That's a little high, but then again, it's a made-up problem.

Note: Problem 2 part (b) requires that you do problem 1 first.

2. **[10] Hubble's law gives a simple relationship between distance and velocity. For this problem, you will assume (i) Hubble's Law is exact, and (ii) the velocity does not change; *i.e.*, if something is currently moving at 100 km/s, it always was moving at that speed.**
   (a) **[5] Assuming constant speed, find a simple formula for how long ago some distant object would have left us t_0, given its current speed. Note that the result does *not* depend on the distance, only on Hubble's constant H_0. This time is called the *Hubble time*, and is a fair estimate of the age of the universe.**

An object at distance d has a velocity $v = H_0 d$. If it left us a time t_0 ago, and is moving at velocity v, then it will have travelled a distance $d = vt_0$. Substituting the latter equation into the former, we have

$$v = H_0 d = H_0 v t_0,$$
$$H_0 t_0 = 1,$$
$$t_0 = H_0^{-1}$$

   (b) **[5] Estimate the Hubble time in Gyr, assuming the Hubble constant you found in problem 1 is correct. How does this compare with our estimate of the age of the oldest stars, 13 ± 1 Gyr?**

This is just a units conversion problem:

$$t_0 = \frac{1}{H_0} = \frac{\text{s}\cdot\text{Mpc}}{72.26\ \text{km}} \cdot \frac{10^6\ \text{pc}}{\text{Mpc}} \cdot \frac{\text{km}}{10^3\ \text{m}} \cdot \frac{3.085\times 10^{16}\ \text{m}}{\text{pc}} \cdot \frac{\text{yr}}{3.156\times 10^7\ \text{s}} \cdot \frac{\text{Gyr}}{10^9\ \text{yr}} = 13.53\ \text{Gyr}$$

The numbers match extremely well, since we would expect the age of the Universe to be a bit longer than the age of the oldest stars.

Graduate problem: Only do this problem if you are in PHY 610

3. **[15] In class, we showed that the age of the universe is given in general by the formula**

$$t_0 = H_0^{-1} \int_0^1 \frac{dx}{\sqrt{\Omega/x + 1 - \Omega}}$$

Complete this integral in closed form. You will probably have to do three cases separately: $\Omega < 1$, $\Omega > 1$, and $\Omega = 1$.

We first rewrite this as $t_0 H_0 = \int_0^1 \sqrt{x}\,dx \big/ \sqrt{\Omega + (1-\Omega)x}$. The case $\Omega = 1$ is by far the easiest, as no change of variable is required. We find $t_0 H_0 = \int_0^1 \sqrt{x}\,dx = \frac{2}{3} x^{3/2}\Big|_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$.

For $\Omega > 1$, we can make the trigonometric substitution $x = \dfrac{\Omega}{\Omega - 1}\sin^2\theta$, which then yields

$$t_0 H_0 = \left(\frac{\Omega}{\Omega-1}\right)^{3/2} \int_0^{\sin^{-1}\sqrt{\frac{\Omega-1}{\Omega}}} \frac{2\sin^2\theta\cos\theta d\theta}{\sqrt{\Omega - \Omega\sin^2\theta}} = \frac{\Omega}{(\Omega-1)^{3/2}} \int_0^{\sin^{-1}\sqrt{\frac{\Omega-1}{\Omega}}} 2\sin^2\theta d\theta$$

$$= \frac{\Omega}{(\Omega-1)^{3/2}} \left[\theta - \tfrac{1}{2}\sin(2\theta)\right]_0^{\sin^{-1}\sqrt{\frac{\Omega-1}{\Omega}}} = \frac{\Omega}{(\Omega-1)^{3/2}} \left[\theta - \sin\theta\cos\theta\right]_0^{\sin^{-1}\sqrt{\frac{\Omega-1}{\Omega}}}$$

$$= \frac{\Omega}{(\Omega-1)^{3/2}} \left[\sin^{-1}\sqrt{\frac{\Omega-1}{\Omega}} - \sqrt{\frac{\Omega-1}{\Omega}}\sqrt{\frac{1}{\Omega}}\right] = \frac{1}{\Omega-1}\left[\frac{\Omega}{\sqrt{\Omega-1}}\tan^{-1}\sqrt{\Omega-1} - 1\right].$$

For $\Omega > 1$, we can make the trigonometric substitution $x = \dfrac{\Omega}{1-\Omega}\sinh^2\theta$, which then yields

$$t_0 H_0 = \left(\frac{\Omega}{1-\Omega}\right)^{3/2} \int_0^{\sinh^{-1}\sqrt{\frac{1-\Omega}{\Omega}}} \frac{2\sinh^2\theta\cos\theta d\theta}{\sqrt{\Omega\sinh^2\theta + \Omega}} = \frac{\Omega}{(1-\Omega)^{3/2}} \int_0^{\sinh^{-1}\sqrt{\frac{1-\Omega}{\Omega}}} 2\sinh^2\theta d\theta$$

$$= \frac{\Omega}{(1-\Omega)^{3/2}} \left[\tfrac{1}{2}\sinh(2\theta) - \theta\right]_0^{\sinh^{-1}\sqrt{\frac{1-\Omega}{\Omega}}} = \frac{\Omega}{(1-\Omega)^{3/2}} \left[\sinh\theta\cosh\theta - \theta\right]_0^{\sinh^{-1}\sqrt{\frac{1-\Omega}{\Omega}}}$$

$$= \frac{\Omega}{(1-\Omega)^{3/2}} \left[\sqrt{\frac{1-\Omega}{\Omega}}\sqrt{\frac{1}{\Omega}} - \sinh^{-1}\sqrt{\frac{1-\Omega}{\Omega}} - \right] = \frac{1}{1-\Omega}\left[1 - \frac{\Omega}{\sqrt{1-\Omega}}\tanh^{-1}\sqrt{1-\Omega}\right].$$

In summary, we have

$$t_0 H_0 = \begin{cases} \dfrac{1}{\Omega-1}\left(\dfrac{\Omega}{\sqrt{\Omega-1}}\tan^{-1}\sqrt{\Omega-1} - 1\right) & \text{if } \Omega > 1, \\ \dfrac{1}{1-\Omega}\left(1 - \dfrac{\Omega}{\sqrt{1-\Omega}}\tanh^{-1}\sqrt{1-\Omega}\right) & \text{if } \Omega < 1, \\ \dfrac{2}{3} & \text{if } \Omega = 1. \end{cases}$$