Physics 310/610 – Cosmology

Solution Set D

Reference	Observed
6562.7 Å	5265.8 Å
4861.3 Å	4701.6 Å
4340.5 Å	4443.0 Å
4101.7 Å	4300.4 Å
	4212.7 Å
	4154.5 Å

1. [15] The table at right shows those spectral lines of hydrogen that are in the range 4000-7000 Å (left column) and the observed spectral lines of a distant star (right column).

(a) [5] One possibility is that the first observed spectral line is a blue-shifted version of the first reference line. If this were the case, then other lines on the right would have the same ratio λ_0/λ. Convince yourself that that this is not the case, in other words, the other spectral lines don't match up.

The observed spectral lines, or at least some of them, should match the reference spectral lines, or at least some of them. They will, however, be Doppler shifted, but the amount of shifting should be the same for each of them. For example, the first observed wavelength could be a blue shifted version of the reference wavelength. We note these are in the ratio $1+z=\lambda_0/\lambda=5265.8/6562.7=0.8024$. However, if this is the case, we should see other observed lines that are a factor of 0.8024 shorter than the reference lines; for example, at $4861.3\times0.8024=3900.7\,\text{Å}$ and $4340.5\times0.8024=3482.8\,\text{Å}$. But these don't correspond to any of the lines.

(b) [5] Another possibility is that the first observed spectral line is a red-shifted version of the *second* reference line. If this is the case, then other lines on the right would have the same ratio λ_0/λ. Convince yourself that that this is the case, in other words, the other spectral lines do match up.

In this case we find the ratio is $1+z=\lambda_0/\lambda=5265.8/4861.3=1.0832$. If this were the case, then we should see observed lines at $1.0832\times4340.5=4701.7\,\text{Å}$, $1.0832\times4101.7\,\text{Å}=4443.0\,\text{Å}$ and $1.0832\times6562.7\,\text{Å}=7108.8\,\text{Å}$. The first two are pretty much exactly right. The $7108.8\ \text{Å}$ is missing because (as stated in the problem) we have only included reference lines in the range $4000-7000\ \text{Å}$.

(c) [5] Find the red-shift z and the velocity v_r of the star.

Since $1+z=1.08321$,. So we have $z=0.08321$. Since this is not extremely close to one, we should probably use the full relativistic formula, $1+z=\sqrt{(1+v_r/c)/(1-v_r/c)}$, to get the velocity

$$(1+z)^2(1-v_r/c)=1+v_r/c,$$
$$(1+z)^2-1=\left[(1+z)^2+1\right](v_r/c),$$

$$\frac{v_r}{c} = \frac{(1+z)^2 - 1}{(1+z)^2 + 1} = \frac{1.0832^2 - 1}{1.0832^2 + 1} = 0.07976,$$

$$v_r = 0.07976\left(2.998 \times 10^8 \text{ m/s}\right) = 23{,}910 \text{ km/s}.$$

2. [15] In class I gave an approximate formula for the luminosity of a star, namely $L = L_\odot \left(M/M_\odot\right)^{3.5}$, where *M* is the mass of the star (confusingly, *M* is also used for the absolute magnitude). You may want to present your answers to this question in the form of a table.

(a) [2] Work out the luminosity, in terms of solar luminosities, for stars of mass 0.1, 0.3, 1, 3, 10, and 30 solar masses

(b) [5] Assume they are placed at a distance of 10 pc from the Earth. What would be their apparent brightness *F*, in W/m²?

The solar luminosities are trivial by direct substitution. The brightness is given by

$$F = \frac{L}{4\pi d^2} = \frac{L_\odot}{4\pi d^2}\left(\frac{M}{M_\odot}\right)^{3.5} = \frac{3.839 \times 10^{26} \text{ W}}{4\pi\left(10 \times 3.086 \times 10^{16} \text{ m}\right)^2}\left(\frac{M}{M_\odot}\right)^{3.5} = \left(3.208 \times 10^{-10} \text{ W/m}^2\right)\left(\frac{M}{M_\odot}\right)^{3.5}$$

Both are included in the tables below

(c) [5] Find their apparent magnitude *m* at this distance. What is the absolute magnitude *M* of these stars?

The apparent magnitude is related to the brightness by

$$m = 2.5\log\left(\frac{2.52 \times 10^{-8} \text{ W/m}^2}{F}\right) = 2.5\log\left[\frac{2.52 \times 10^{-8} \text{ W/m}^2}{3.208 \times 10^{-10} \text{ W/m}^2}\left(\frac{M}{M_\odot}\right)^{-3.5}\right]$$

$$= 2.5\log(78.6) - 8.75\log\left(M/M_\odot\right) = 4.74 - 8.75\log\left(M/M_\odot\right)$$

It is now a straightforward matter to find the apparent magnitude of these stars. However, since 10 pc is the magic distance at which the magnitude is defined, this is the same as the absolute magnitude. Hence in the table I have labeled it as *M*, the absolute magnitude.

$\frac{M}{M_\odot}$	$\frac{L}{L_\odot}$	10 pc F (W/m²)	10 pc M	1 kpc m
0.1	0.00032	1.014×10^{-13}	13.49	23.49
0.3	0.015	4.744×10^{-12}	9.32	19.32
1	1.00	3.208×10^{-10}	4.74	14.74
3	47	1.500×10^{-8}	0.57	10.57
10	3,200	1.014×10^{-6}	– 4.01	5.99
30	148,000	4.744×10^{-5}	– 8.18	1.82

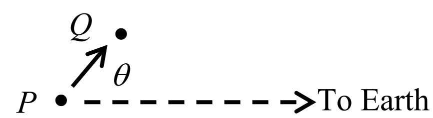
(d) [3] Suppose these same stars were brought to a distance of 1 kpc instead. How would their apparent and absolute magnitudes change?

The absolute magnitude doesn't change. However, the apparent magnitude is related to the absolute magnitude and the distance by

$$m - M = 5\log d - 5 = 5\log(1000) - 5 = 15 - 5 = 10$$

Hence the apparent magnitude at 1 kpc is simply the absolute magnitude plus ten.

Graduate Problem – Do if you are in PHY 610



3. [20] Suppose an object is moving at an angle θ compared to straight towards you at a speed *v* that is less than but comparable to the speed of light *c*. (Although this problem involves relativistic velocities, there is no relativity in this problem)

(a) [5] Assume the object starts at a point *P* and moves to a point *Q* for a time *t*. How much closer is it to you at time *t*? How much delay Δt is there between when you receive light from *P* and light from *Q*?

It is moving a distance *vt* in this amount of time. Because it is coming towards you at an angle θ, the distance it comes closer will be $vt\cos\theta$.

Now, the time between the two signals has two causes: one is just because there was a time lag *t* between the two signals, and another because the second point is closer by $vt\cos\theta$. Since the speed of light is *c*, this means that the second signal will have an advantage time of $vt\cos\theta/c$. So the total time difference is $\Delta t = t - vt\cos\theta/c$.

(b) [5] What is the transverse distance d_T that the object moves during this time? Find the apparent transverse velocity $v_{aT} = d_T/\Delta t$ as a function of *v* and θ.

The transverse distance it will come is just $d_T = vt\sin\theta$. We then have the apparent transverse velocity as

$$v_{aT} = \frac{d_T}{\Delta t} = \frac{tv\sin\theta}{t - tv\cos\theta/c} = \frac{v\sin\theta}{1 - v\cos\theta/c}.$$

As we shall see, for sufficiently high velocities, this can exceed the speed of light.

(c) [10] Find the maximum value $v_{\max}$ of v_{aT} as a function of θ for fixed *v*. Note that it is larger than *v*; i.e., $v_{\max} > v$. What is the smallest value of *v* such that we can have $v_{\max} \geq c$?

It is pretty easy to find the maximum by simply taking the derivative and setting it to zero. So we have

$$0 = \frac{d}{d\theta} v_{aT} = \frac{\left(1 - v\cos\theta/c\right)v\cos\theta - v\sin\theta\left(v\sin\theta/c\right)}{\left(1 - v\cos\theta/c\right)^2} = \frac{v\left(\cos\theta - v/c\right)}{\left(1 - v\cos\theta/c\right)^2}.$$

It is trivial to see that this has solution $\cos\theta = v/c$. We then use $\sin^2\theta + \cos^2\theta = 1$ to conclude that $\sin\theta = \sqrt{1 - v^2/c^2}$. Substituting this into our formula for v_{aT}, we have

$$v_{\max} = \frac{v\sin\theta_{\max}}{1 - v\cos\theta_{\max}/c} = \frac{v\sqrt{1 - v^2/c^2}}{1 - v^2/c^2} = \frac{v}{\sqrt{1 - v^2/c^2}}.$$

For low velocities, this is approximately *v*. It reaches the speed of light when $v_{\max} = c$, or when

$$c = \frac{v}{\sqrt{1 - v^2/c^2}},$$
$$\sqrt{1 - v^2/c^2} = v/c,$$
$$1 - v^2/c^2 = v^2/c^2,$$
$$v^2/c^2 = \tfrac{1}{2}$$
$$v = \tfrac{1}{\sqrt{2}}c.$$

So we can have apparent transverse speeds greater than the speed of light if $v \geq \tfrac{1}{\sqrt{2}}c$.