Physics 310 – Cosmology

# Solution Set V

1. **[12] This question concerns the relative strength of electric and gravitational forces.**
   **(a) [3] Write a formula for the gravitational force between two electrons. Find the ratio of the gravitational force to the Coulomb force, $F = ke^2/r^2$, where *k*is Coulomb's constant and *e* is the electron charge (which you can go look up). Show that the ratio is constant and evaluate it.**

The gravitational force between two particles is $F_g = Gm^2/r^2$, and the ratio of these forces is

$$\frac{F_g}{F_E} = \frac{Gm^2/r^2}{ke^2/r^2} = \frac{Gm^2}{ke^2} = \frac{\left(6.674\times10^{-11}\ \text{m}^3/\text{kg/s}^2\right)\left(9.1094\times10^{-31}\ \text{kg}\right)^2}{\left(8.9876\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(1.602\times10^{-19}\ \text{C}\right)^2} = 2.40\times10^{-43}$$

As we can see, at conventional energies (such as the electron mass), gravity is far weaker than other forces such as the electric force.

   **(b) [5] For very energetic particles, the mass *m* for the electron is replaced by $E/c^2$. For what energy *E* in GeV will the gravitational force between a pair of electrons be as strong as the electric force?**

The electric force does not change, because it depends only on the electric charge. The two forces will match when

$$1 = \frac{F_g}{F_E} = \frac{Gm^2/r^2}{ke^2/r^2} = \frac{G\left(E/c^2\right)^2}{ke^2},$$

$$E/c^2 = \sqrt{ke^2/G},$$

$$E = c^2\sqrt{ke^2/G} = \left(2.998\times10^{8}\ \text{m/s}\right)^2\sqrt{\frac{\left(8.9876\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(1.602\times10^{-19}\ \text{C}\right)^2}{6.674\times10^{-11}\ \text{m}^3/\text{kg/s}^2}}$$

$$= \frac{1.671\times10^{5}\ \text{J}}{1.602\times10^{-19}\ \text{J/eV}} = 1.043\times10^{27}\ \text{eV} = 1.043\times10^{18}\ \text{GeV}.$$

**(c) [4] A typical energy of a particle at temperature *T* is given by $3k_BT$. What is $k_BT$ when gravity is the same strength as the other forces? How old is the universe at this time? Assume the universe is radiation dominated at this time, with $g_{\text{eff}} = 200$ for definiteness.**

Obviously, $k_BT = \frac{1}{3}E = 3.48\times10^{17}\ \text{GeV}$. We then simply substitute this into the formula for the age of the universe.

$$t = \frac{2.42\ \text{s}}{\sqrt{g_{\text{eff}}}}\left(\frac{\text{MeV}}{k_BT}\right)^2 = \frac{2.42\ \text{s}}{\sqrt{200}}\left(\frac{\text{MeV}}{3.48\times10^{20}\ \text{MeV}}\right)^2 = 1.41\times10^{-42}\ \text{s}$$

**2. [18] Before inflation, the universe could be quite small, at least in principle. It might be no bigger than *ct*, where *t* is the age of the universe, and *c* the speed of light.**

**(a) [4] Assuming this early universe is radiation dominated and has $\Omega = 1$, find a simple relationship between the age *t* and the energy density $\rho$. It is helpful to use the Friedmann equation, and use the fact that in a flat radiation-dominated universe, $Ht = \frac{1}{2}$.**

The Friedman equation says that $H^2 = \frac{8}{3}\pi G\rho - kc^2/a^2$, but if the universe is flat (or close to it) the second term is negligible, so this simplifies to $H^2 = \frac{8}{3}\pi G\rho$. Replacing *H* with $1/2t$, we have

$$\frac{1}{4t^2} = \frac{8}{3}\pi G\rho,$$
$$\rho = \frac{3}{32\pi G t^2}.$$

**(b) [4] Show that the mass *M* of the whole universe can be written as $M = c^3t/8G$. Assume the universe is a sphere of radius *ct*.**

We assume the universe is a sphere of radius *ct*. We multiply the volume times the density to get

$$M = V\rho = \tfrac{4}{3}\pi R^3\rho = \frac{4\pi(ct)^3}{3}\cdot\frac{3}{32\pi G t^2} = \frac{c^3t}{8G}$$

as advertised.

**(c) [4] The total energy of the universe is $Mc^2$. The age of the universe is $t$. According to quantum mechanics, it is possible to create an energy $E$ for a time $t$ from nothingness provided $Et \le \frac{1}{2}\hbar$. Find a simple formula for a time $t$ such that $Et = \frac{1}{2}\hbar$.**

The total energy is $E = Mc^2 = c^5 t / 8G$. If $Et = \frac{1}{2}\hbar$, then

$$\frac{c^5 t}{8G} t = \frac{\hbar}{2},$$
$$t^2 = \frac{4G\hbar}{c^5},$$
$$t = 2\sqrt{G\hbar / c^5}\,.$$

**(d) [6] Evaluate the time found in part (c). It is possible that the universe was spontaneously created from nothing at this time (the ultimate free lunch). Assuming it immediately goes into a radiation dominated era with $g_{\text{eff}} = 200$, find the temperature at this time.**

The numerical value is just

$$t = 2\sqrt{G\hbar / c^5} = 2\sqrt{\frac{\left(6.674\times 10^{-11}\ \text{m}^3/\text{kg/s}^2\right)\left(1.0546\times 10^{-34}\ \text{J}\cdot\text{s}\right)}{\left(2.998\times 10^{8}\ \text{m/s}\right)^5}} = 1.08\times 10^{-43}\ \text{s}$$

We now use the formula relating time and temperature, namely,

$$1.08\times 10^{-43}\ \text{s} = t = \frac{2.42\ \text{s}}{\sqrt{g_{\text{eff}}}}\left(\frac{\text{MeV}}{k_B T}\right)^2 = \left(0.1711\ \text{s}\right)\left(\frac{\text{MeV}}{k_B T}\right)^2,$$
$$\left(\frac{k_B T}{\text{MeV}}\right)^2 = \frac{1.08\times 10^{-43}\ \text{s}}{0.1711\ \text{s}} = 1.58\times 10^{-42},$$
$$k_B T = \sqrt{1.58\times 10^{-42}}\ \text{MeV} = 1.26\times 10^{21}\ \text{MeV}$$

I *intended* you to calculate it this way (or to write $1.26\times 10^{18}\ \text{GeV}$), but since it is worded as the *temperature* we should divide by $k_B$ to give

$$T = 1.46\times 10^{31}\ \text{K}\,.$$