

Black-Hole Binaries in Quasi-Equilibrium

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Wake Forest University

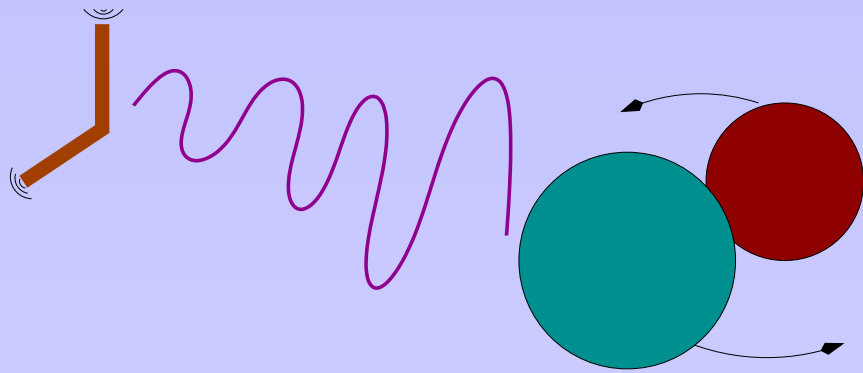
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Abstract

A formalism for constructing initial data representing black-hole binaries in quasi-equilibrium is developed. If each black hole is assumed to be in quasi-equilibrium, then a complete set of boundary conditions for *all* initial data variables can be developed. This formalism should allow for the construction of completely general quasi-equilibrium black hole binary initial data.

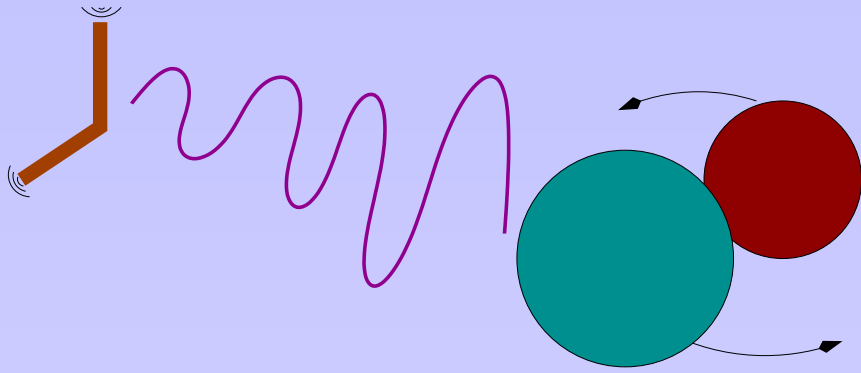
[Related LANL preprint. . .](#)

Motivation



- Black hole binaries are among the most likely sources for early detection with LIGO, VIRGO, GEO, . . .
- Available computed waveforms should increase chance of detecting collision events.

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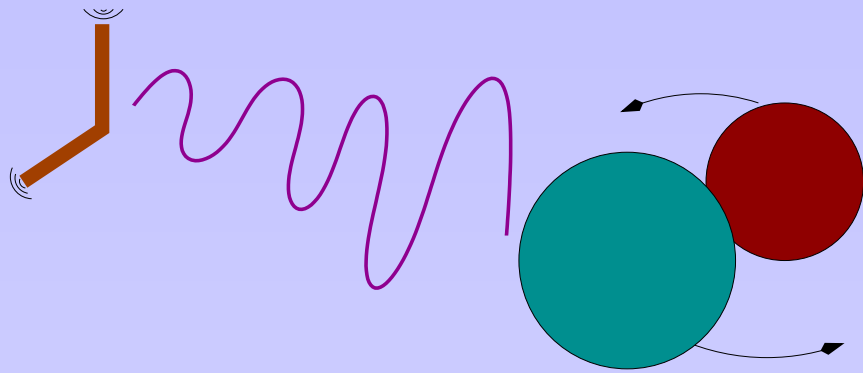


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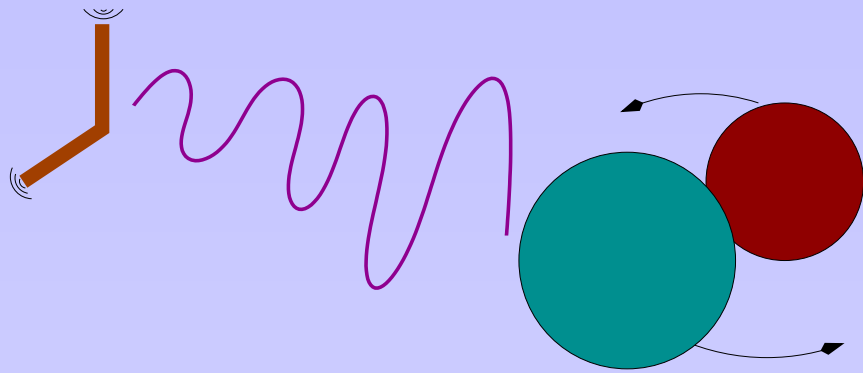


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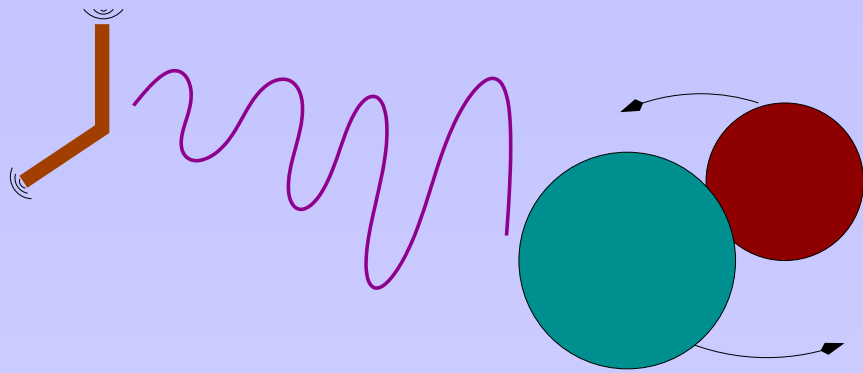


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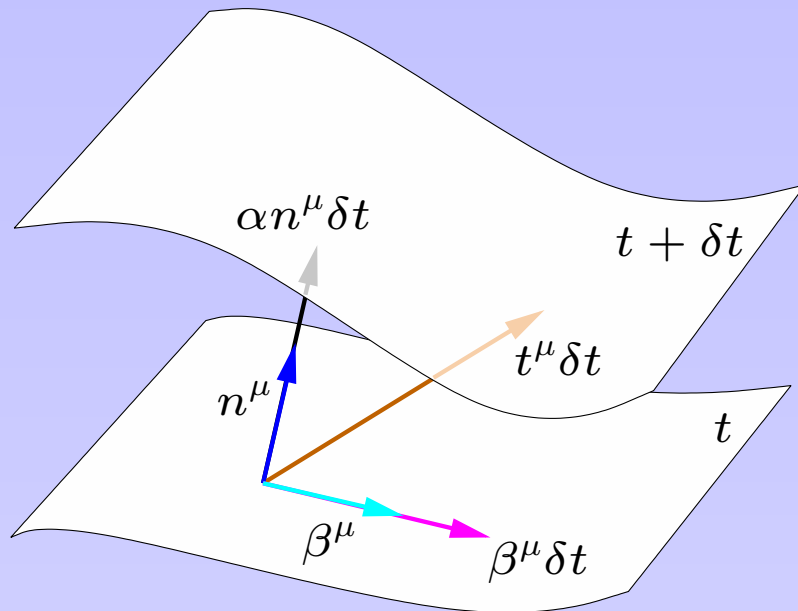


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- If the orbit is nearly circular (*quasi-circular*) then there is a *corotating reference frame* in which the binary appears to be at rest.
- ★ Quasi-equilibrium gives us a *physical condition* to guide us in fixing boundary conditions and data that is not otherwise constrained.

The 3 + 1 Decomposition



Lapse : α

Spatial metric : γ_{ij}

Shift vector : β^i

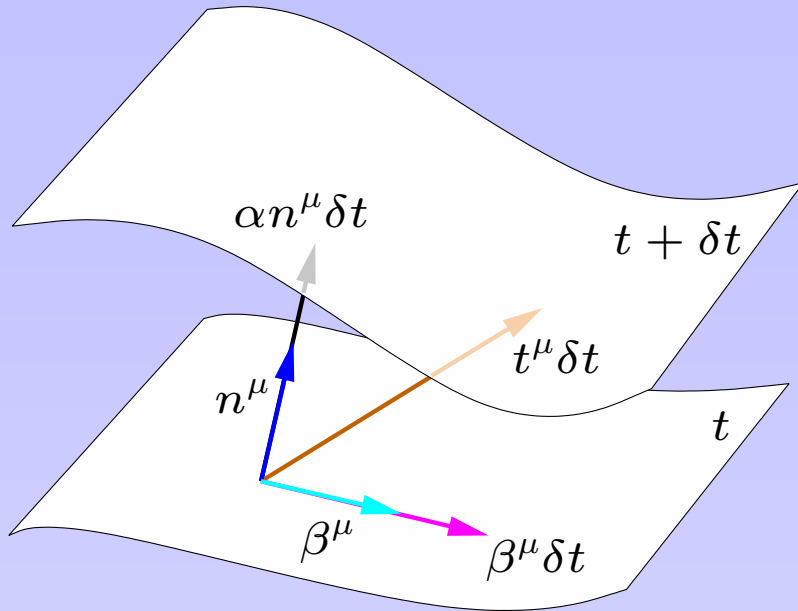
Extrinsic Curvature : K_{ij}

Time vector : $t^\mu = \alpha n^\mu + \beta^\mu$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \quad K_{\mu\nu} = -\frac{1}{2} \gamma_\mu^\alpha \gamma_\nu^\beta \mathcal{L}_n g_{\alpha\beta}$$

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Constraint equations

$$\bar{R} + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

$$\bar{\nabla}_j (K^{ij} - \gamma^{ij}K) = 8\pi j^i$$

$$S_{\mu\nu} \equiv \gamma_\mu^\alpha \gamma_\nu^\beta T_{\alpha\beta}$$

$$j_\mu \equiv -\gamma_\mu^\nu n^\alpha T_{\nu\alpha}$$

$$\rho \equiv n^\mu n^\nu T_{\mu\nu}$$

$$T_{\mu\nu} = S_{\mu\nu} + 2n_{(\mu}j_{\nu)} + n_\mu n_\nu \rho$$

Evolution equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i$$

$$\partial_t K_{ij} = -\bar{\nabla}_i \bar{\nabla}_j \alpha + \alpha \left[\bar{R}_{ij} - 2K_{il}K_j^\ell + K K_{ij} - 8\pi S_{ij} + 4\pi \gamma_{ij}(S - \rho) \right]$$

$$+ \beta^\ell \bar{\nabla}_\ell K_{ij} + K_{il} \bar{\nabla}_j \beta^\ell + K_{jl} \bar{\nabla}_i \beta^\ell$$

Degrees of Freedom

Kinematical variables

- Lapse α : 1 degree of freedom
- Shift β^i : 3 degrees of freedom

Initial-data variables

- Metric γ_{ij} : 6 degrees of freedom
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Decomposition of initial-data variables[5] (*Conformal TT decomp*)

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \left\{ \begin{array}{l} \psi : 1 \text{ constrained DOF} \\ \tilde{\gamma}_{ij} : \left\{ \begin{array}{l} 3 \text{ spatial gauge DOF} \\ 2 \text{ dynamical DOF} \end{array} \right. \end{array} \right.$$

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$$K^{ij} = \psi^{-10} \left[(\tilde{\mathbb{L}}X)^{ij} + \tilde{Q}^{ij} \right] + \frac{1}{3} \gamma^{ij} K \left\{ \begin{array}{l} \tilde{Q}^{ij} : \left\{ \begin{array}{l} \tilde{\nabla}_j \tilde{Q}^{ij} = \tilde{Q}^i_i = 0 \\ 2 \text{ dynamical DOF} \end{array} \right. \\ X^i : \left\{ \begin{array}{l} (\tilde{\mathbb{L}}X)^{ij} \equiv \tilde{\nabla}^i X^j + \tilde{\nabla}^j X^i - \frac{1}{3} \tilde{\gamma}^{ij} \tilde{\nabla}_k X^k \\ 3 \text{ constrained DOF} \end{array} \right. \\ K : 1 \text{ temporal gauge DOF} \end{array} \right.$$

Specifying Initial Data

Freely specified degrees of freedom

$\tilde{\gamma}_{ij} \Leftarrow \begin{cases} (2) \text{ initial dynamical ("wave") content} \\ (3) \text{ initial spatial gauge choices} \end{cases}$

$\tilde{Q}^{ij} \Leftarrow (2) \text{ initial dynamical ("wave") content}$

$K \Leftarrow (1) \text{ initial temporal gauge choice}$

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Constrained degrees of freedom

$$\psi \Leftarrow \begin{cases} (1) \text{ Hamiltonian constraint} \\ \tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2\pi \psi^5 \rho \end{cases}$$

$$X^i \Leftarrow \begin{cases} (3) \text{ Momentum constraint} \\ \tilde{\Delta}_{\mathbb{L}} X^i = \frac{2}{3} \psi^6 \tilde{\nabla}^i K + 8\pi \psi^{10} j^i \end{cases}$$

$$\tilde{A}^{ij} \equiv (\tilde{\mathbb{L}}X)^{ij} + \tilde{Q}^{ij}$$

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Boundary conditions

The constraints form a set of 4 coupled nonlinear PDEs for (ψ, X^i) that require the specification of boundary conditions at spatial infinity and any interior boundaries.

“Traditional” Black-Hole Data

Conformal flatness and maximal slicing

$$\left. \begin{array}{l} \tilde{\gamma}_{ij} = f_{ij} \text{ (flat)} \\ \tilde{Q}^{ij} = 0 \\ K = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \tilde{\Delta}_{\mathbb{L}} X^i = 0 \Rightarrow \\ \tilde{\nabla}^2 \psi + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0 \end{array} \right.$$

Bowen-York solution[3]
Analytic solutions for \tilde{A}^{ij}

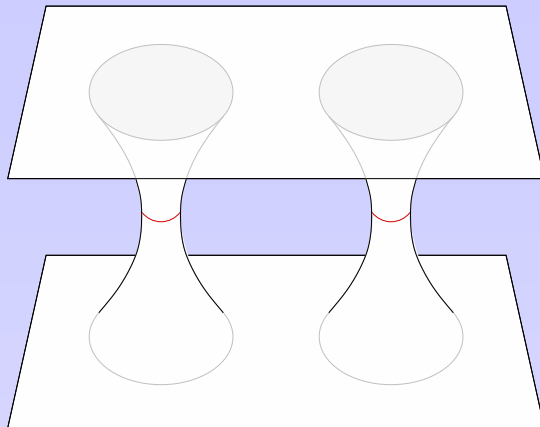
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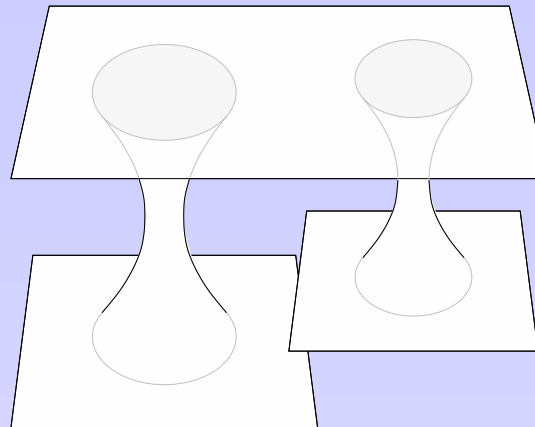
Three general solution schemes

Conformal Imaging-[6]



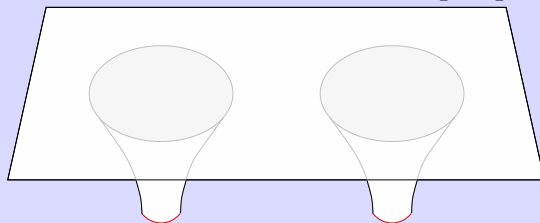
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Puncture Method-[4]



No inner-BC:
singular
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Apparent Horizon BC-[11]



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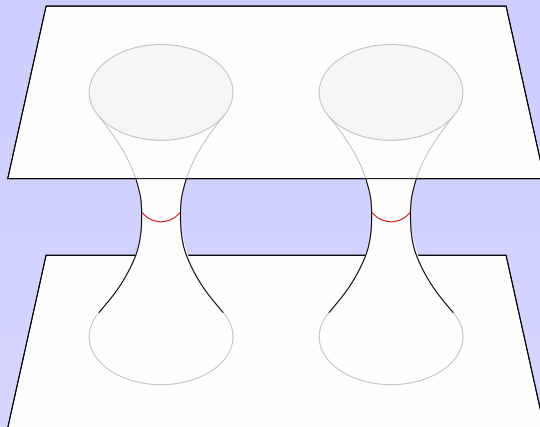
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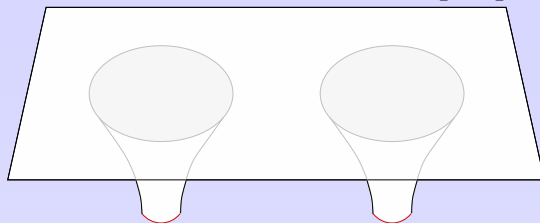
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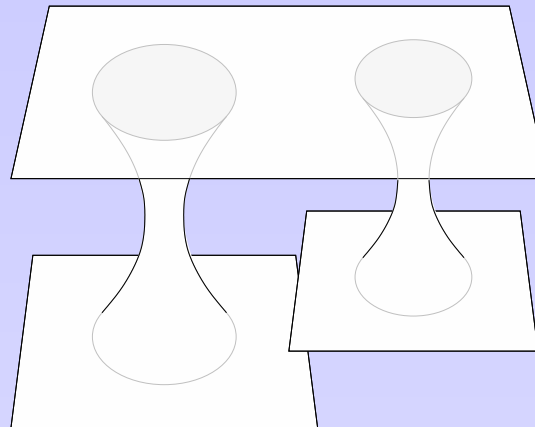
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All methods can produce very general configurations of multiple black holes, but are fundamentally limited by choices for $\tilde{\gamma}_{ij}$ and Bowen-York \tilde{A}^{ij} .

“Better” Black-Hole Data

What is wrong with “traditional” BH initial data?

- Results disagree with PN predictions for black holes in quasi-circular orbits.
- There is no control of the initial “wave” content.
- Spinning holes are not represented well.

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How do we construct improved BH initial data?

We must carefully choose the

- initial dynamical degrees of freedom [in $\tilde{\gamma}_{ij}$ and \tilde{Q}^{ij}]
- initial temporal and spatial gauge degrees of freedom [in $\tilde{\gamma}_{ij}$ and K]
- boundary conditions on the constrained degrees of freedom [in ψ and X^i]

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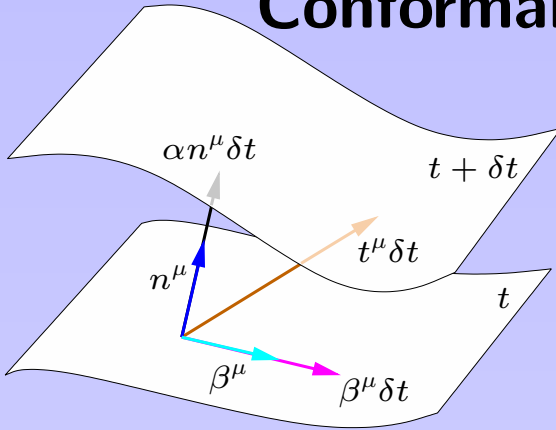
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so as to conform to the desired physical content of the initial data.

- For black holes in quasi-circular orbits, we can use the principle of *quasi-equilibrium* to guide our choices.
- *Quasi-equilibrium* is a *dynamical* concept and we can simplify our task by choosing a decomposition of the initial-data variables that has connections to dynamics.

Conformal Thin-Sandwich Decomposition[13]



$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$$

$$K^{ij} = \frac{\psi^{-10}}{2\tilde{\alpha}} \left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right] + \frac{1}{3}\gamma^{ij} K \begin{cases} \tilde{u}_{ij} \equiv \partial_t \tilde{\gamma}_{ij} & (\tilde{u}_i^i = 0) \\ \tilde{\alpha} \equiv \psi^{-6} \alpha \end{cases}$$

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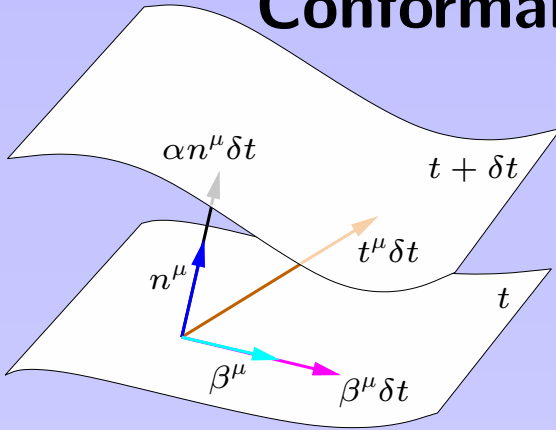
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Constrained vars : ψ and β^i

Freely specified : $\tilde{\gamma}_{ij}$, \tilde{u}^{ij} , K , and $\tilde{\alpha}$

\tilde{u}^{ij} and β^i have a simple physical interpretation, unlike \tilde{Q}^{ij} and X^i .

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$$\text{Quasi-equilibrium} \Rightarrow \begin{cases} \tilde{u}^{ij} = 0 \\ \partial_t K = 0 \text{ (Const. on } \alpha) \end{cases}$$

Const. Tr(K) eqn. $\tilde{\nabla}^2(\alpha\psi) - \alpha \left[\frac{1}{8}\psi \tilde{R} + \frac{5}{12}\psi^5 K^2 + \frac{7}{8}\psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} + 2\pi\psi^5 K(\rho + 2S) \right] = \psi^5 \beta^i \tilde{\nabla}_i K$

Equations of Quasi-Equilibrium

Ham. & Mom. const.
eqns. from Conf. TS
+ Const. Tr(K) eqn. } \Rightarrow Eqs. of
Quasi-Equilibrium

With $\tilde{\gamma}_{ij} = f_{ij}$, $\tilde{u}^{ij} = 0$, and $K = 0$, these equations have been widely used to construct binary neutron star initial data [1, 10, 2, 12].

Binary neutron star initial data require:

- boundary conditions at infinity compatible with asymptotic flatness and corotation.

$$\psi|_{r \rightarrow \infty} = 1 \quad \beta^i|_{r \rightarrow \infty} = \Omega \left(\frac{\partial}{\partial \phi} \right)^i \quad \alpha|_{r \rightarrow \infty} = 1$$

- compatible solution of the equations of hydrostatic equilibrium. ($\Rightarrow \Omega$)

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Binary black hole initial data require:

- a means for choosing the angular velocity of the orbit Ω .

★ *with excision*, inner boundary conditions are needed for ψ , β^i , and $\tilde{\alpha}$.

Gourgoulhon, Grandclément, & Bonazzola [8, 9]: Black-hole binaries with $\tilde{\gamma}_{ij} = f_{ij}$, $\tilde{u}^{ij} = 0$, $K = 0$, “inversion-symmetry”, and “Killing-horizon” conditions on the excision boundaries.

“Solutions” require *constraint violating* regularity condition imposed on inner boundaries!

Constructing Regular Binary Black Hole QE ID

Why does the GGB approach fail?

- Inversion-symmetry demands $\tilde{\alpha} = 0$ & $K = 0$ on the inner boundary.

- It is hard to move beyond $\tilde{\gamma}_{ij} = f_{ij}$.

$$\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right]$$

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- Find a method that allows for general choices of $\tilde{\gamma}_{ij}$ & K .
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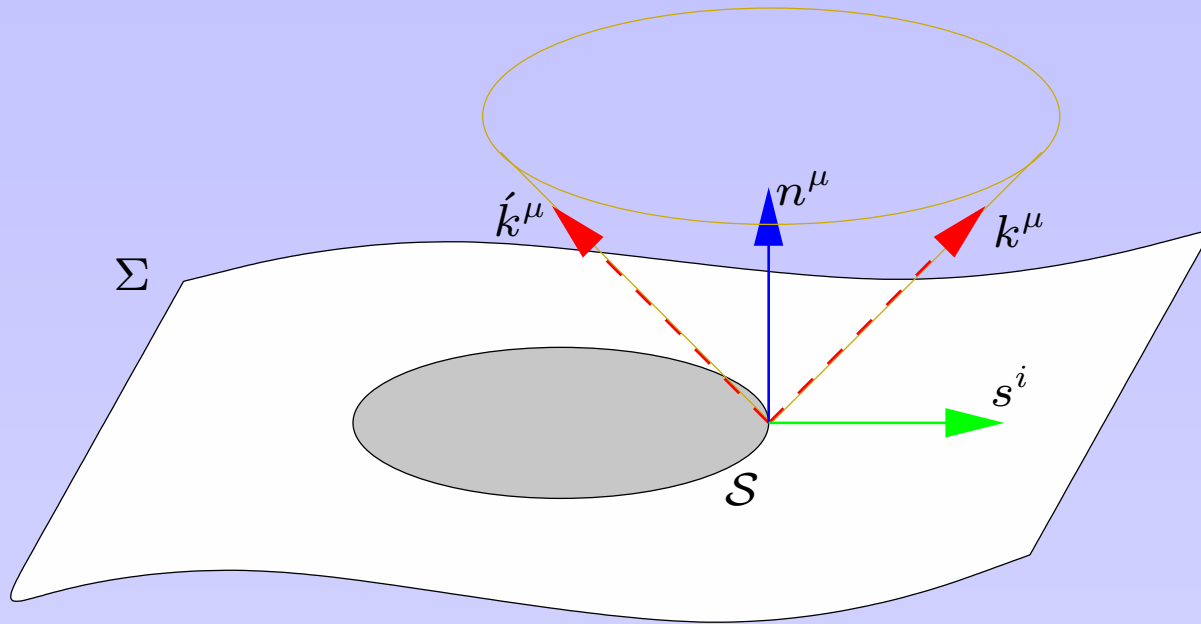
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Approach

- Demand that the excision (*inner*) boundary be an *apparent horizon*.
- Demand that the apparent horizon be in quasi-equilibrium.

The Inner Boundary



$$s_i \equiv \frac{\bar{\nabla}_i \tau}{|\bar{\nabla} \tau|}$$

$$h_{ij} \equiv \gamma_{ij} - s_i s_j$$

$$k^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu + s^\mu)$$

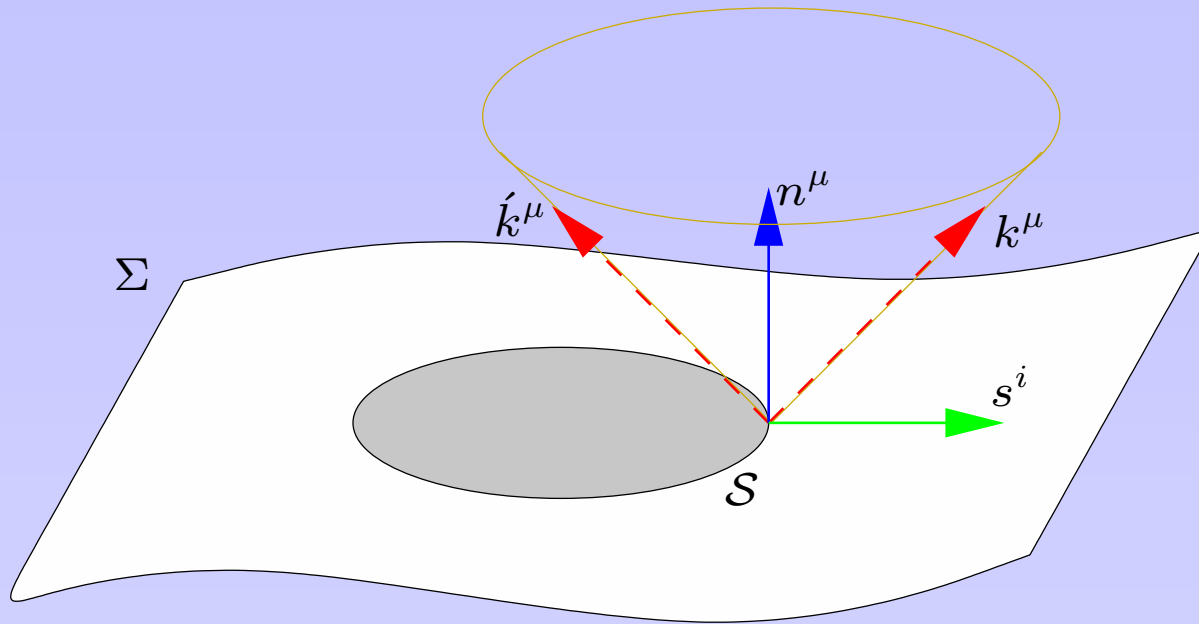
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Extrinsic curvature of S embedded in
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$$\hat{\Sigma}_{\mu\nu} \equiv -\frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_{\hat{k}} g_{\alpha\beta}$$

Extrinsic curvature of S embedded in Σ

$$H_{ij} \equiv -\frac{1}{2} h_i^k h_j^\ell \mathcal{L}_s \gamma_{kl}$$

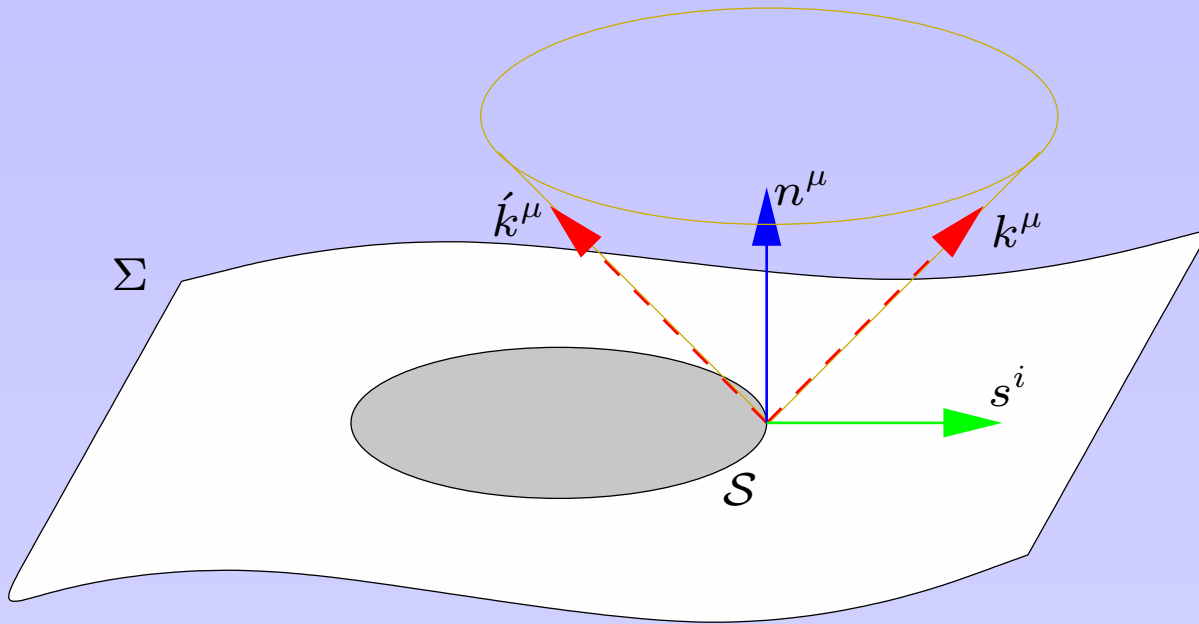
Projections of K_{ij} onto S

$$J_{ij} \equiv h_i^k h_j^\ell K_{kl}$$

$$J_i \equiv h_i^k s^\ell K_{kl}$$

$$J \equiv h^{ij} J_{ij} = h^{ij} K_{ij}$$

The Inner Boundary



$$s_i \equiv \frac{\bar{\nabla}_i \tau}{|\bar{\nabla} \tau|}$$

$$h_{ij} \equiv \gamma_{ij} - s_i s_j$$

$$k^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu + s^\mu)$$

$$\hat{k}^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu - s^\mu)$$

Extrinsic curvature of S embedded in spacetime

$$\Sigma_{\mu\nu} \equiv -\frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_k g_{\alpha\beta}$$

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Extrinsic curvature of S embedded in Σ

$$H_{ij} \equiv -\frac{1}{2} h_i^k h_j^\ell \mathcal{L}_s \gamma_{kl}$$

$$\Sigma_{ij} = \frac{1}{\sqrt{2}} (J_{ij} + H_{ij})$$

$$\hat{\Sigma}_{ij} = \frac{1}{\sqrt{2}} (J_{ij} - H_{ij})$$

Projections of K_{ij} onto S

$$J_{ij} \equiv h_i^k h_j^\ell K_{kl}$$

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Expansion of null rays

$$\sigma \equiv h^{ij} \Sigma_{ij} = \frac{1}{\sqrt{2}} (J + H)$$

$$\hat{\sigma} \equiv h^{ij} \hat{\Sigma}_{ij} = \frac{1}{\sqrt{2}} (J - H)$$

Shear of null rays

$$\sigma_{ij} \equiv \Sigma_{ij} - \frac{1}{2} h_{ij} \sigma$$

$$\hat{\sigma}_{ij} \equiv \hat{\Sigma}_{ij} - \frac{1}{2} h_{ij} \hat{\sigma}$$

AH and QE Conditions on the Inner Boundary

The quasi-equilibrium inner boundary conditions start with the following assumptions:

1. The inner boundary \mathcal{S} is a (MOTS):
marginally outer-trapped surface

$$\rightarrow \sigma = 0$$

2. The inner boundary \mathcal{S} doesn't move:

$$\rightarrow \mathcal{L}_\zeta \tau = 0 \text{ and } \hat{\nabla}_i \mathcal{L}_\zeta \tau \equiv h_i^j \bar{\nabla}_j \mathcal{L}_\zeta \tau = 0$$

$$t^\mu = \alpha n^\mu + \beta^\mu \quad \zeta^\mu \equiv \alpha n^\mu + \beta_\perp s^\mu$$

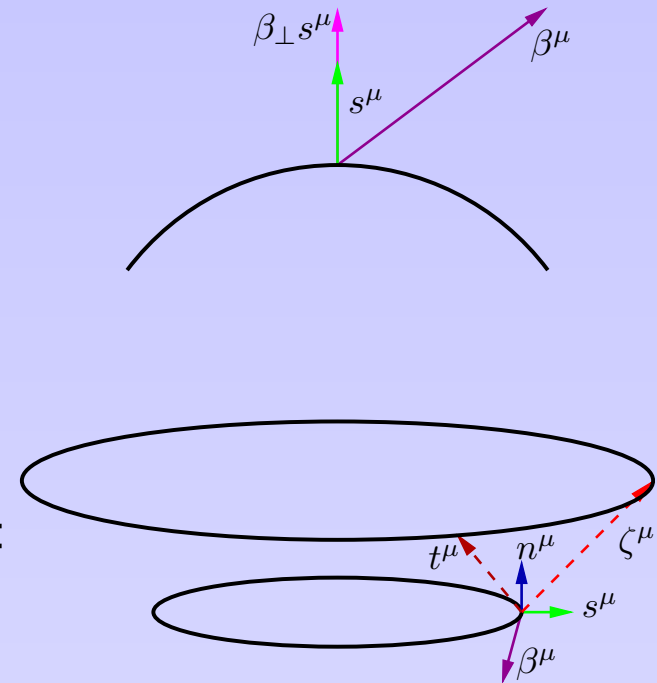
$$\beta_\perp \equiv \beta^i s_i$$

3. The inner boundary \mathcal{S} remains a MOTS[7]:

$$\rightarrow \mathcal{L}_\zeta \sigma = 0 \text{ and } \mathcal{L}_\zeta \acute{\sigma} = 0$$

4. The horizons are in quasi-equilibrium:

$$\rightarrow \sigma_{ij} = 0 \text{ and no matter is on } \mathcal{S}$$



Evolution of the Expansions

$$\begin{aligned}
 \mathcal{L}_\zeta \sigma &= \frac{1}{\sqrt{2}} \left[\sigma \left(\sigma + \frac{1}{2} \acute{\sigma} - \frac{1}{\sqrt{2}} K \right) + \mathcal{E} \right] (\beta_\perp + \alpha) \\
 &+ \frac{1}{\sqrt{2}} \left[\sigma \left(\frac{1}{2} \sigma - \frac{1}{2} \acute{\sigma} - \frac{1}{\sqrt{2}} K \right) + \mathcal{D} + 8\pi T_{\mu\nu} k^\mu \acute{k}^\nu \right] (\beta_\perp - \alpha) \\
 &+ \sigma s^i \bar{\nabla}_i \alpha, \\
 \mathcal{L}_\zeta \acute{\sigma} &= -\frac{1}{\sqrt{2}} \left[\acute{\sigma} \left(\acute{\sigma} + \frac{1}{2} \sigma - \frac{1}{\sqrt{2}} K \right) + \acute{\mathcal{E}} \right] (\beta_\perp - \alpha) \\
 &- \frac{1}{\sqrt{2}} \left[\acute{\sigma} \left(\frac{1}{2} \acute{\sigma} - \frac{1}{2} \sigma - \frac{1}{\sqrt{2}} K \right) + \acute{\mathcal{D}} + 8\pi T_{\mu\nu} k^\mu \acute{k}^\nu \right] (\beta_\perp + \alpha) \\
 &- \acute{\sigma} s^i \bar{\nabla}_i \alpha,
 \end{aligned}$$

$$\mathcal{D} \equiv h^{ij} (\hat{\nabla}_i + J_i) (\hat{\nabla}_j + J_j) - \frac{1}{2} \hat{R}$$

$$\acute{\mathcal{D}} \equiv h^{ij} (\hat{\nabla}_i - J_i) (\hat{\nabla}_j - J_j) - \frac{1}{2} \hat{R}$$

$$\mathcal{E} \equiv \sigma_{ij} \sigma^{ij} + 8\pi T_{\mu\nu} k^\mu k^\nu$$

$$\acute{\mathcal{E}} \equiv \acute{\sigma}_{ij} \acute{\sigma}^{ij} + 8\pi T_{\mu\nu} \acute{k}^\mu \acute{k}^\nu$$

Incorporates the constraint and evolution equations of GR, the Gauss–Codazzi–Ricci equations governing the embedding of \mathcal{S} in the spatial hypersurface, and the demand that \mathcal{S} remain at a constant coordinate location. *These equations incorporate no assumption of quasi-equilibrium.*

Terms that vanish because we demand \mathcal{S} be a MOTS, remain a MOTS, or because we demand the horizon to be in equilibrium.

AH/Quasi-Equilibrium Boundary Conditions

$$\begin{aligned} \sigma &= 0 \\ 0 &= \mathcal{D}(\beta_{\perp} - \alpha), \\ \acute{\sigma} s^i \bar{\nabla}_i \alpha &= -\frac{1}{\sqrt{2}} \left[\acute{\sigma} \left(\acute{\sigma} - \frac{1}{\sqrt{2}} K \right) + \acute{\sigma}_{ij} \acute{\sigma}^{ij} \right] (\beta_{\perp} - \alpha) \\ &\quad - \frac{1}{\sqrt{2}} \left[\acute{\sigma} \left(\frac{1}{2} \acute{\sigma} - \frac{1}{\sqrt{2}} K \right) + \acute{\mathcal{D}} \right] (\beta_{\perp} + \alpha). \end{aligned} \Rightarrow$$

$$\begin{aligned} \tilde{s}^k \tilde{\nabla}_k \ln \psi &= -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J) \\ \beta^i &= \alpha \psi^{-2} \tilde{s}^i + B_{\parallel}^i \\ J \tilde{s}^i \tilde{\nabla}_i \alpha &= -\psi^2 (J^2 - JK + \tilde{\mathcal{D}}) \alpha \end{aligned}$$

$$h_{ij} \equiv \psi^4 \tilde{h}_{ij}$$

$$s^i \equiv \psi^{-2} \tilde{s}^i$$

$$B_{\parallel}^i s_i = 0$$

$$\tilde{\mathcal{D}} \equiv \psi^{-4} [\tilde{h}^{ij} (\check{\nabla}_i - J_i) (\check{\nabla}_j - J_j) - \frac{1}{2} \check{R} + 2 \check{\nabla}^2 \ln \psi]$$

$[\check{\nabla} \ \& \ \check{R} \text{ are compatible with } \tilde{h}_{ij}]$

The conditions of quasi-equilibrium yield boundary conditions for 3 of the 5 constrained variables $(\psi, \alpha, \beta_{\perp})$. The remaining two conditions are contained in the definition of β_{\parallel}^i . This freedom is necessary to prescribe the spin of the black hole.

Defining the Spin of the Black Hole

The spin parameters β_{\parallel}^i can be defined by demanding that the MOTS be a *Killing horizon*. The time vector associated with quasi-equilibrium in the corotating frame must be null, forming the null generators of the horizon.

$$k^{\mu} \propto (n^{\mu} + s^{\mu}) \implies k^{\mu} = [1, \alpha s^i - \beta^i]$$

This vector k^{μ} is null for any choice of β^i .

In the frame where a black hole is not spinning, the null time vector has components $t^{\mu} = [1, \vec{0}]$.

Corotating Holes

Corotating holes are at rest in the corotating frame, where we must pose boundary conditions. So,

$$k^{\mu} = [1, \alpha s^i - \beta^i] = [1, \vec{0}]$$

Thus we find

$$\beta^i = \alpha s^i \implies \beta_{\parallel}^i = 0$$

Irrotational Holes

Irrotational holes are at rest in the inertial frame. With the time vectors in the inertial and corotating frames related by

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} + \Omega \frac{\partial}{\partial \phi}$$

$$k^{\mu} = [1, \alpha s^i - \beta^i] = [1, -\Omega(\partial/\partial\phi)^i]$$

Thus we find

$$\beta^i = \alpha s^i + \Omega \left(\frac{\partial}{\partial \phi} \right)^i \implies \beta_{\parallel}^i = \Omega \left(\frac{\partial}{\partial \phi} \right)^i$$

Summary of QE Formalism

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \gamma^{ij} K \quad \tilde{A}^{ij} = \frac{\psi^6}{2\alpha} (\tilde{\mathbb{L}}\beta)^{ij} \quad \partial_t \tilde{\gamma}_{ij} = 0$$

$$\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0$$

$$\tilde{\Delta}_{\mathbb{L}} \beta^i - (\tilde{\mathbb{L}}\beta)^{ij} \tilde{\nabla}_j \ln \alpha \psi^{-6} = \frac{4}{3} \alpha \tilde{\nabla}^i K$$

$$\tilde{\nabla}^2 (\alpha \psi) - (\alpha \psi) \left[\frac{1}{8} \tilde{R} + \frac{5}{12} \psi^4 K^2 + \frac{7}{8} \psi^{-8} \tilde{A}_{ij} \tilde{A}^{ij} \right] = \psi^5 \beta^i \tilde{\nabla}_i K \quad \partial_t K = 0$$

$$\tilde{s}^k \tilde{\nabla}_k \ln \psi|_S = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J)|_S \quad \sigma = 0$$

$$\beta^i|_S = \begin{cases} \alpha \psi^{-2} \tilde{s}^i|_S & \text{corotation} \\ \alpha \psi^{-2} \tilde{s}^i|_S + \Omega \tilde{h}_j^i \left(\frac{\partial}{\partial \phi} \right)^j \Big|_S & \text{irrotation} \end{cases} \quad \begin{matrix} \mathcal{L}_\zeta \sigma = 0 \\ \sigma_{ij} = 0 \end{matrix}$$

$$J \tilde{s}^i \tilde{\nabla}_i \alpha|_S = -\psi^2 (J^2 - JK + \tilde{\mathcal{D}}) \alpha|_S \quad \mathcal{L}_\zeta \sigma = 0$$

$$\begin{aligned} \psi|_{r \rightarrow \infty} &= 1 \\ \beta^i|_{r \rightarrow \infty} &= \Omega \left(\frac{\partial}{\partial \phi} \right)^i \\ \alpha|_{r \rightarrow \infty} &= 1 \end{aligned}$$

The only remaining freedom in the system is the choice of the orbital angular velocity, the initial spatial and temporal gauge, and the initial dynamical (“wave”) content found in Ω , $\tilde{\gamma}_{ij}$ and K .

The Orbital Angular Velocity

- For a given choice of $\tilde{\gamma}_{ij}$ and K , we are still left with a family of solutions parameterized by the orbital angular velocity Ω .
- Except for the case of a single spinning black hole, it is not reasonable to expect more than one value of Ω to correspond to a system in quasi-equilibrium.

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GGB[8, 9] have suggested a way to pick the quasi-equilibrium value of Ω :

Ω is chosen as the value for which the ADM energy E_{ADM} equals the Komar mass M_{K} .

Komar
mass

$$M_{\text{K}} = \frac{1}{4\pi} \oint_{\infty} \gamma^{ij} (\bar{\nabla}_i \alpha - \beta^k K_{ik}) d^2 S_j$$

Acceptable definition of the mass
only for stationary spacetimes.

ADM
energy

$$E_{\text{ADM}} = \frac{1}{16\pi} \oint_{\infty} \gamma^{ij} \bar{\nabla}_k (\mathcal{G}_i^k - \delta_i^k \mathcal{G}) d^2 S_j$$

Acceptable definition of the mass
for arbitrary spacetimes.

$$\mathcal{G}_{ij} \equiv \gamma_{ij} - f_{ij}$$

Remaining Issues

- Are there solutions satisfying the AH/QE boundary conditions?
 - Stationary BH solutions.
 - Time-symmetric & inversion-symmetric BH solutions.
 - Others?

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 - Use “superposition” of Kerr?
 - Use post-Newtonian metrics (no radiation reaction)?
- Do numerical solutions exist?
 - H. Pfeiffer and I are working on it.
 - The lapse boundary condition is *difficult*.
 - We are currently seeing some “strange” behavior which is either a bug or something very interesting.

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