

Toward Astrophysical Black-Hole Binaries

Gregory B. Cook

Wake Forest University

Mar. 29, 2002

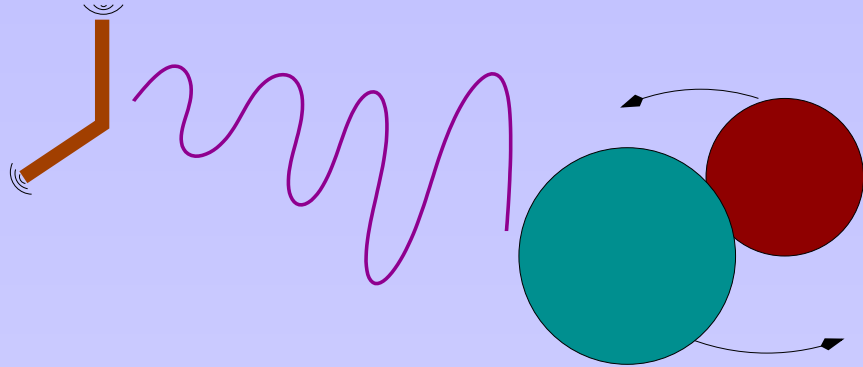
Abstract

A formalism for constructing initial data representing black-hole binaries in quasi-equilibrium is developed. If each black hole is assumed to be in quasi-equilibrium, then a complete set of boundary conditions for *all* initial data variables can be developed. This formalism should allow for the construction of completely general quasi-equilibrium black hole binary initial data.

[5] [Related LANL preprint. . .](#)

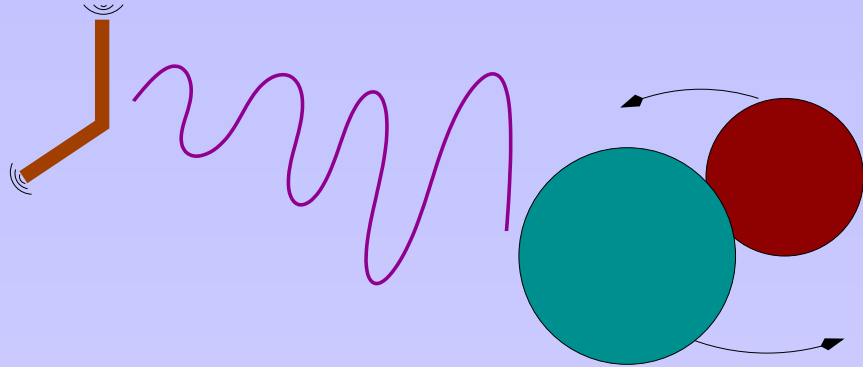
Collaborators: Harald Pfeiffer & Saul Teukolsky (Cornell)

Motivation



- How do we go about constructing *improved* initial-data sets that more accurately represent astrophysical compact binary systems?
- How do we define astrophysically realistic data?

Motivation

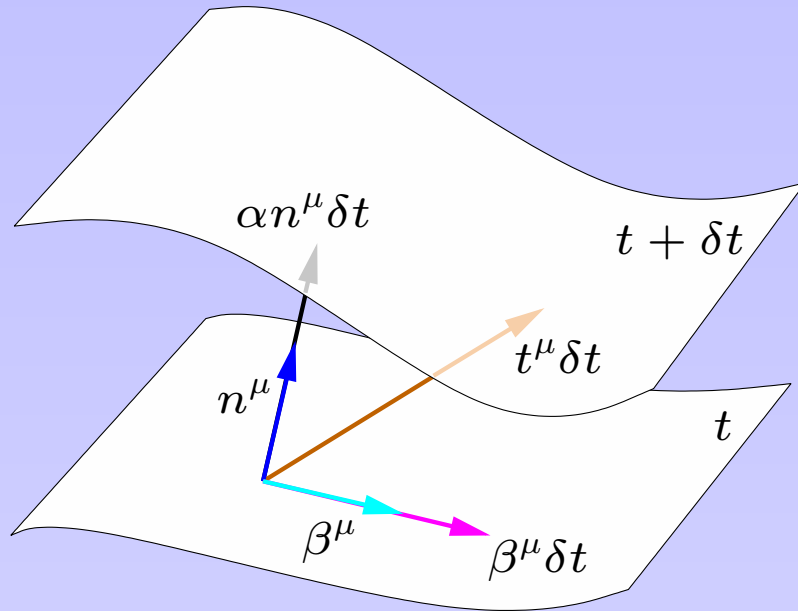


- How do we go about constructing *improved* initial-data sets that more accurately represent astrophysical compact binary systems?
- How do we define astrophysically realistic data?

Focus Issues

- Which decomposition of the constraints will be used?
- How do we choose boundary conditions so that the constraints are well-posed and yield solutions with the desired physical content?
- What choices for the spatial and temporal gauge are compatible with the desired physical content?
- How do we fix the remaining freely specifiable data so as to yield the desired physical content?

The 3 + 1 Decomposition



Lapse : α

Spatial metric : γ_{ij}

Shift vector : β^i

Extrinsic Curvature : K_{ij}

Time vector : $t^\mu = \alpha n^\mu + \beta^\mu$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \quad K_{\mu\nu} = -\frac{1}{2}\gamma_\mu^\alpha \gamma_\nu^\beta \mathcal{L}_n g_{\alpha\beta}$$

Constraint equations

$$\bar{R} + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

$$\bar{\nabla}_j (K^{ij} - \gamma^{ij}K) = 8\pi j^i$$

$$S_{\mu\nu} \equiv \gamma_\mu^\alpha \gamma_\nu^\beta T_{\alpha\beta}$$

$$j_\mu \equiv -\gamma_\mu^\nu n^\alpha T_{\nu\alpha}$$

$$\rho \equiv n^\mu n^\nu T_{\mu\nu}$$

$$T_{\mu\nu} = S_{\mu\nu} + 2n_{(\mu}j_{\nu)} + n_\mu n_\nu \rho$$

Evolution equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i$$

$$\partial_t K_{ij} = -\bar{\nabla}_i \bar{\nabla}_j \alpha + \alpha \left[\bar{R}_{ij} - 2K_{il}K_j^\ell + K K_{ij} - 8\pi S_{ij} + 4\pi \gamma_{ij}(S - \rho) \right]$$

$$+ \beta^\ell \bar{\nabla}_\ell K_{ij} + K_{il} \bar{\nabla}_j \beta^\ell + K_{jl} \bar{\nabla}_i \beta^\ell$$

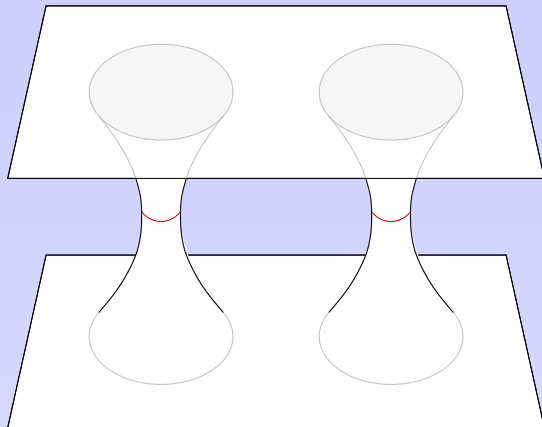
“Traditional” Black-Hole Data

Conformal flatness and maximal slicing

$$\left. \begin{array}{l} \tilde{\gamma}_{ij} = f_{ij} \text{ (flat)} \\ K = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \tilde{\Delta}_{\perp} X^i = 0 \\ \tilde{\nabla}^2 \psi + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0 \end{array} \right. \Rightarrow \begin{array}{l} \text{Bowen-York solution [3]} \\ \text{Analytic solutions for } \tilde{A}^{ij} \\ \text{(conformal tracefree extrinsic curvature)} \end{array}$$

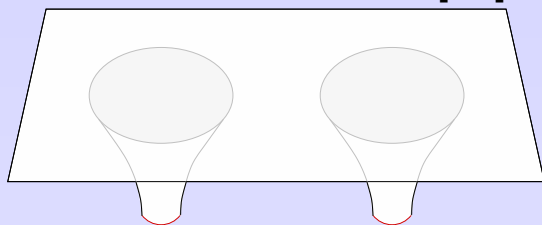
Three general solution schemes

Conformal Imaging-[6]



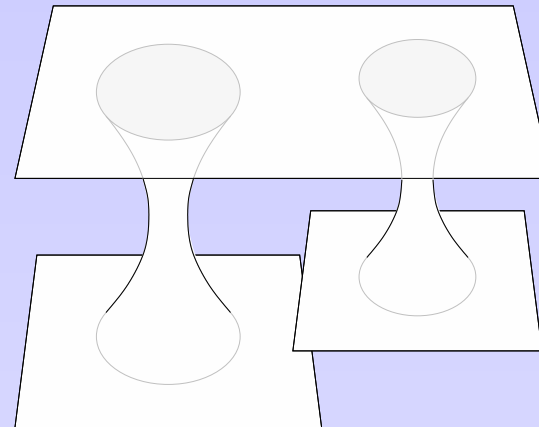
Inversion
symmetry
inner-BC

Apparent Horizon BC-[11]



Apparent
horizon
inner-BC

Puncture Method-[4]



No inner-BC:
singular
behavior
factored out

All methods can produce very general configurations of multiple black holes, but are fundamentally limited by choices for $\tilde{\gamma}_{ij}$ and Bowen-York \tilde{A}^{ij} .

“Better” Black-Hole Data

What is wrong with “traditional” BH initial data?

- Results disagree with PN predictions for black holes in quasi-circular orbits.
- There is no control of the initial “wave” content.
- Spinning holes are not represented well.

“Better” Black-Hole Data

What is wrong with “traditional” BH initial data?

- Results disagree with PN predictions for black holes in quasi-circular orbits.
- There is no control of the initial “wave” content.
- Spinning holes are not represented well.

How do we construct improved BH initial data?

We must carefully choose the

- initial dynamical degrees of freedom [in $\tilde{\gamma}_{ij}$ and \tilde{A}_{TT}^{ij}]
- initial temporal and spatial gauge degrees of freedom [in $\tilde{\gamma}_{ij}$ and K]
- boundary conditions on the constrained degrees of freedom [in ψ and X^i]

so as to conform to the desired physical content of the initial data.

“Better” Black-Hole Data

What is wrong with “traditional” BH initial data?

- Results disagree with PN predictions for black holes in quasi-circular orbits.
- There is no control of the initial “wave” content.
- Spinning holes are not represented well.

How do we construct improved BH initial data?

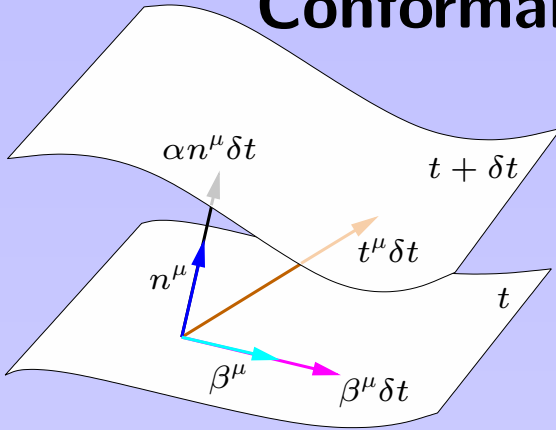
We must carefully choose the

- initial dynamical degrees of freedom [in $\tilde{\gamma}_{ij}$ and \tilde{A}_{TT}^{ij}]
- initial temporal and spatial gauge degrees of freedom [in $\tilde{\gamma}_{ij}$ and K]
- boundary conditions on the constrained degrees of freedom [in ψ and X^i]

so as to conform to the desired physical content of the initial data.

- For black holes in quasi-circular orbits, we can use the principle of *quasi-equilibrium* to guide our choices.
- *Quasi-equilibrium* is a *dynamical* concept and we can simplify our task by choosing a decomposition of the initial-data variables that has connections to dynamics.

Conformal Thin-Sandwich Decomposition[13]



$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$$

$$K^{ij} = \frac{\psi^{-10}}{2\tilde{\alpha}} \left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right] + \frac{1}{3}\gamma^{ij} K \begin{cases} \tilde{u}_{ij} \equiv \partial_t \tilde{\gamma}_{ij} & (\tilde{u}_i^i = 0) \\ \tilde{\alpha} \equiv \psi^{-6} \alpha \end{cases}$$

Hamiltonian Const. $\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2\pi \psi^5 \rho$

Momentum Const. $\tilde{\Delta}_{\mathbb{L}} \beta^i - (\tilde{\mathbb{L}}\beta)^{ij} \tilde{\nabla}_j \tilde{\alpha} = \frac{4}{3} \tilde{\alpha} \psi^6 \tilde{\nabla}^i K + \tilde{\alpha} \tilde{\nabla}_j \left(\frac{1}{\tilde{\alpha}} \tilde{u}^{ij} \right) + 16\pi \tilde{\alpha} \psi^{10} j^i$

$$\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right]$$

Constrained vars : ψ and β^i

Freely specified : $\tilde{\gamma}_{ij}$, \tilde{u}^{ij} , K , and $\tilde{\alpha}$

\tilde{u}^{ij} and β^i have a simple physical interpretation, unlike \tilde{A}_{TT}^{ij} and X^i .

$$\text{Quasi-equilibrium} \Rightarrow \begin{cases} \tilde{u}^{ij} = 0 \\ \partial_t K = 0 \text{ (Const. on } \alpha) \end{cases}$$

Const. Tr(K) eqn. $\tilde{\nabla}^2(\alpha\psi) - \alpha \left[\frac{1}{8} \psi \tilde{R} + \frac{5}{12} \psi^5 K^2 + \frac{7}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} + 2\pi \psi^5 K(\rho + 2S) \right] = \psi^5 \beta^i \tilde{\nabla}_i K$

Equations of Quasi-Equilibrium

Ham. & Mom. const.
eqns. from Conf. TS
+ Const. Tr(K) eqn. } \Rightarrow Eqs. of
Quasi-Equilibrium

With $\tilde{\gamma}_{ij} = f_{ij}$, $\tilde{u}^{ij} = 0$, and $K = 0$, these equations have been widely used to construct binary neutron star initial data [1, 10, 2, 12].

Binary neutron star initial data require:

- boundary conditions at infinity compatible with asymptotic flatness and corotation.

$$\psi|_{r \rightarrow \infty} = 1 \quad \beta^i|_{r \rightarrow \infty} = \Omega \left(\frac{\partial}{\partial \phi} \right)^i \quad \alpha|_{r \rightarrow \infty} = 1$$

- compatible solution of the equations of hydrostatic equilibrium. ($\Rightarrow \Omega$)

Equations of Quasi-Equilibrium

Ham. & Mom. const. eqns. from Conf. TS
+ Const. $\text{Tr}(K)$ eqn. } \Rightarrow Eqs. of Quasi-Equilibrium

With $\tilde{\gamma}_{ij} = f_{ij}$, $\tilde{u}^{ij} = 0$, and $K = 0$, these equations have been widely used to construct binary neutron star initial data[1, 10, 2, 12].

Binary neutron star initial data require:

- boundary conditions at infinity compatible with asymptotic flatness and corotation.

$$\psi|_{r \rightarrow \infty} = 1 \quad \beta^i|_{r \rightarrow \infty} = \Omega \left(\frac{\partial}{\partial \phi} \right)^i \quad \alpha|_{r \rightarrow \infty} = 1$$

- compatible solution of the equations of hydrostatic equilibrium. ($\Rightarrow \Omega$)

Binary black hole initial data require:

- a means for choosing the angular velocity of the orbit Ω .
- ★ *with excision*, inner boundary conditions are needed for ψ , β^i , and $\tilde{\alpha}$.

Gourgoulhon, Grandclément, & Bonazzola[8, 9]: Black-hole binaries with $\tilde{\gamma}_{ij} = f_{ij}$, $\tilde{u}^{ij} = 0$, $K = 0$, “inversion-symmetry”, and “Killing-horizon” conditions on the excision boundaries.

“Solutions” require *constraint violating* regularity condition imposed on inner boundaries!

Constructing Regular Binary Black Hole QE ID

Why does the GGB approach have problems?

- Inversion-symmetry demands $\tilde{\alpha} = 0$ & $K = 0$ on the inner boundary.

- It is hard to move beyond $\tilde{\gamma}_{ij} = f_{ij}$.

$$\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right]$$

Constructing Regular Binary Black Hole QE ID

Why does the GGB approach have problems?

- Inversion-symmetry demands $\tilde{\alpha} = 0$ & $K = 0$ on the inner boundary.

- It is hard to move beyond $\tilde{\gamma}_{ij} = f_{ij}$.

$$\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right]$$

How do we proceed?

- Find a method that allows for general choices of $\tilde{\gamma}_{ij}$ & K .
- ★ Eliminate dependence on inversion symmetry by letting the physical condition of quasi-equilibrium dictate the boundary conditions.

Constructing Regular Binary Black Hole QE ID

Why does the GGB approach have problems?

- Inversion-symmetry demands $\tilde{\alpha} = 0$ & $K = 0$ on the inner boundary.

- It is hard to move beyond $\tilde{\gamma}_{ij} = f_{ij}$.

$$\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right]$$

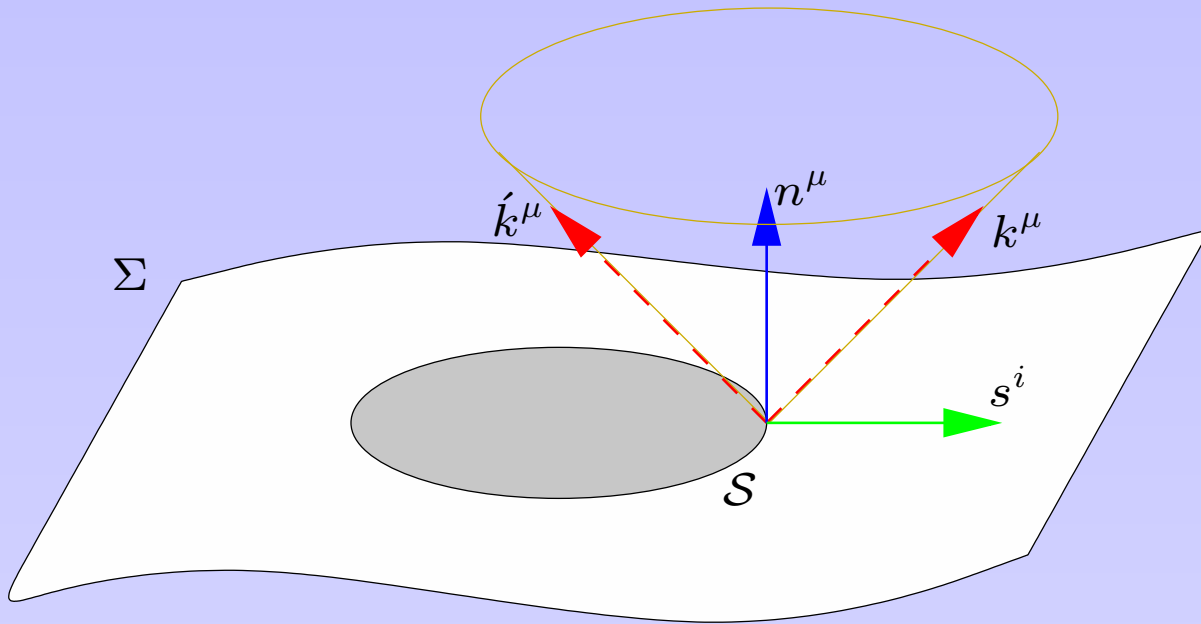
How do we proceed?

- Find a method that allows for general choices of $\tilde{\gamma}_{ij}$ & K .
- ★ Eliminate dependence on inversion symmetry by letting the physical condition of quasi-equilibrium dictate the boundary conditions.

Approach

- Demand that the excision (*inner*) boundary be an *apparent horizon*.
- Demand that the apparent horizon be in quasi-equilibrium.

The Inner Boundary



$$s_i \equiv \frac{\bar{\nabla}_i \tau}{|\bar{\nabla} \tau|}$$

$$h_{ij} \equiv \gamma_{ij} - s_i s_j$$

$$k^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu + s^\mu)$$

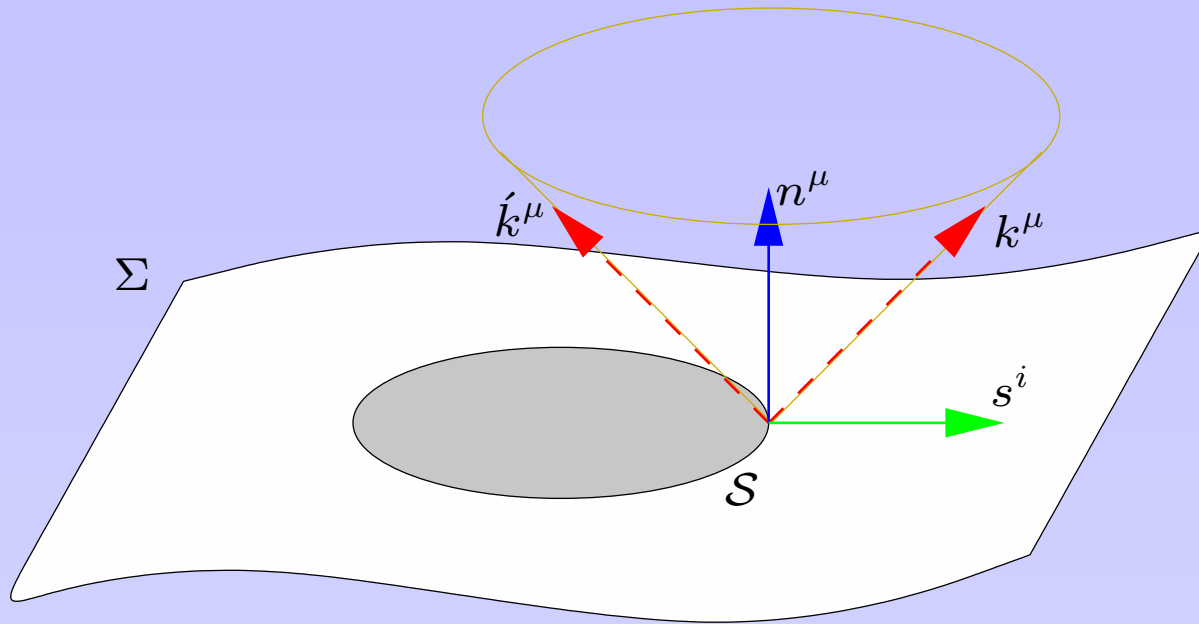
$$\hat{k}^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu - s^\mu)$$

Extrinsic curvature of S embedded in
spacetime

$$\Sigma_{\mu\nu} \equiv -\frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_k g_{\alpha\beta}$$

$$\hat{\Sigma}_{\mu\nu} \equiv -\frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_{\hat{k}} g_{\alpha\beta}$$

The Inner Boundary



$$s_i \equiv \frac{\bar{\nabla}_i \tau}{|\bar{\nabla} \tau|}$$

$$h_{ij} \equiv \gamma_{ij} - s_i s_j$$

$$k^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu + s^\mu)$$

$$\hat{k}^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu - s^\mu)$$

Extrinsic curvature of S embedded in spacetime

$$\Sigma_{\mu\nu} \equiv -\frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_k g_{\alpha\beta}$$

$$\hat{\Sigma}_{\mu\nu} \equiv -\frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_{\hat{k}} g_{\alpha\beta}$$

Extrinsic curvature of S embedded in Σ

$$H_{ij} \equiv -\frac{1}{2} h_i^k h_j^\ell \mathcal{L}_s \gamma_{kl}$$

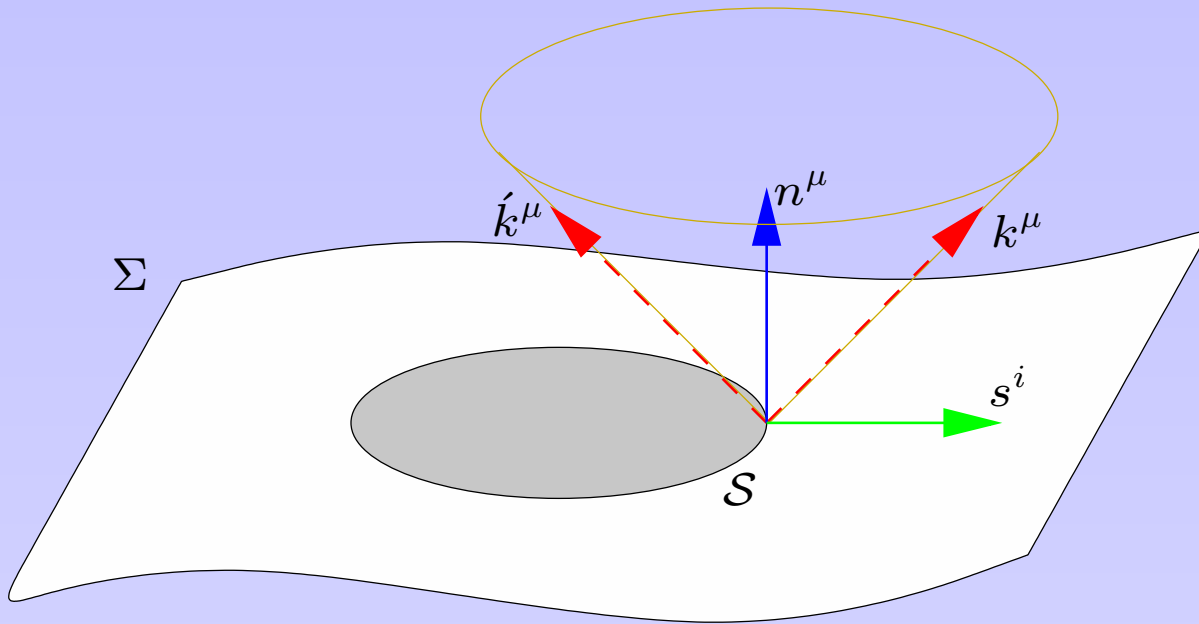
Projections of K_{ij} onto S

$$J_{ij} \equiv h_i^k h_j^\ell K_{kl}$$

$$J_i \equiv h_i^k s^\ell K_{kl}$$

$$J \equiv h^{ij} J_{ij} = h^{ij} K_{ij}$$

The Inner Boundary



$$s_i \equiv \frac{\bar{\nabla}_i \tau}{|\bar{\nabla} \tau|}$$

$$h_{ij} \equiv \gamma_{ij} - s_i s_j$$

$$k^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu + s^\mu)$$

$$\hat{k}^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu - s^\mu)$$

Extrinsic curvature of S embedded in spacetime

$$\Sigma_{\mu\nu} \equiv -\frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_k g_{\alpha\beta}$$

$$\hat{\Sigma}_{\mu\nu} \equiv -\frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_{\hat{k}} g_{\alpha\beta}$$

Extrinsic curvature of S embedded in Σ

$$H_{ij} \equiv -\frac{1}{2} h_i^k h_j^\ell \mathcal{L}_s \gamma_{kl}$$

$$\Sigma_{ij} = \frac{1}{\sqrt{2}} (J_{ij} + H_{ij})$$

$$\hat{\Sigma}_{ij} = \frac{1}{\sqrt{2}} (J_{ij} - H_{ij})$$

Projections of K_{ij} onto S

$$J_{ij} \equiv h_i^k h_j^\ell K_{kl}$$

$$J_i \equiv h_i^k s^\ell K_{kl}$$

$$J \equiv h^{ij} J_{ij} = h^{ij} K_{ij}$$

Expansion of null rays

$$\theta \equiv h^{ij} \Sigma_{ij} = \frac{1}{\sqrt{2}} (J + H)$$

$$\hat{\theta} \equiv h^{ij} \hat{\Sigma}_{ij} = \frac{1}{\sqrt{2}} (J - H)$$

Shear of null rays

$$\sigma_{ij} \equiv \Sigma_{ij} - \frac{1}{2} h_{ij} \theta$$

$$\hat{\sigma}_{ij} \equiv \hat{\Sigma}_{ij} - \frac{1}{2} h_{ij} \hat{\theta}$$

AH and QE Conditions on the Inner Boundary

The quasi-equilibrium inner boundary conditions start with the following assumptions:

1. The inner boundary \mathcal{S} is a (MOTS):
marginally outer-trapped surface

$$\rightarrow \theta = 0$$

2. The inner boundary \mathcal{S} doesn't move:

$$\rightarrow \mathcal{L}_\zeta \tau = 0 \text{ and } \hat{\nabla}_i \mathcal{L}_\zeta \tau \equiv h_i^j \bar{\nabla}_j \mathcal{L}_\zeta \tau = 0$$

$$t^\mu = \alpha n^\mu + \beta^\mu \quad \zeta^\mu \equiv \alpha n^\mu + \beta_\perp s^\mu$$

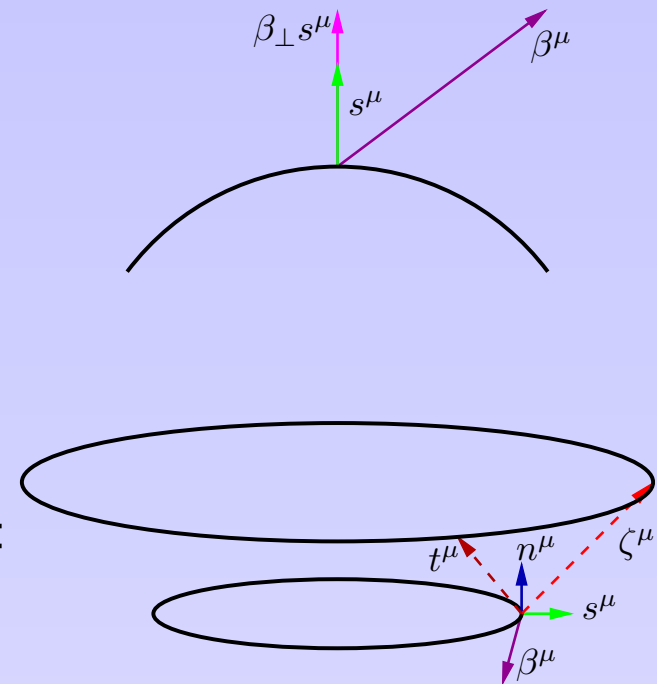
$$\beta_\perp \equiv \beta^i s_i$$

3. The inner boundary \mathcal{S} remains a MOTS[7]:

$$\rightarrow \mathcal{L}_\zeta \theta = 0 \text{ and } \mathcal{L}_\zeta \dot{\theta} = 0$$

4. The horizons are in quasi-equilibrium:

$$\rightarrow \sigma_{ij} = 0 \text{ and no matter is on } \mathcal{S}$$



Evolution of the Expansions

$$\begin{aligned}
 \mathcal{L}_\zeta \theta &= \frac{1}{\sqrt{2}} \left[\theta \left(\theta + \frac{1}{2} \dot{\theta} - \frac{1}{\sqrt{2}} K \right) + \mathcal{E} \right] (\beta_\perp + \alpha) \\
 &+ \frac{1}{\sqrt{2}} \left[\theta \left(\frac{1}{2} \dot{\theta} - \frac{1}{2} \theta - \frac{1}{\sqrt{2}} K \right) + \mathcal{D} + 8\pi T_{\mu\nu} k^\mu \dot{k}^\nu \right] (\beta_\perp - \alpha) \\
 &+ \theta s^i \bar{\nabla}_i \alpha, \\
 \mathcal{L}_\zeta \dot{\theta} &= -\frac{1}{\sqrt{2}} \left[\dot{\theta} \left(\dot{\theta} + \frac{1}{2} \theta - \frac{1}{\sqrt{2}} K \right) + \mathcal{E}' \right] (\beta_\perp - \alpha) \\
 &- \frac{1}{\sqrt{2}} \left[\dot{\theta} \left(\frac{1}{2} \dot{\theta} - \frac{1}{2} \theta - \frac{1}{\sqrt{2}} K \right) + \mathcal{D}' + 8\pi T_{\mu\nu} k^\mu \dot{k}^\nu \right] (\beta_\perp + \alpha) \\
 &- \dot{\theta} s^i \bar{\nabla}_i \alpha,
 \end{aligned}$$

$$\mathcal{D} \equiv h^{ij} (\hat{\nabla}_i + J_i) (\hat{\nabla}_j + J_j) - \frac{1}{2} \hat{R}$$

$$\mathcal{D}' \equiv h^{ij} (\hat{\nabla}_i - J_i) (\hat{\nabla}_j - J_j) - \frac{1}{2} \hat{R}$$

$$\mathcal{E} \equiv \sigma_{ij} \sigma^{ij} + 8\pi T_{\mu\nu} k^\mu k^\nu$$

$$\mathcal{E}' \equiv \dot{\sigma}_{ij} \dot{\sigma}^{ij} + 8\pi T_{\mu\nu} \dot{k}^\mu \dot{k}^\nu$$

Incorporates the constraint and evolution equations of GR, the Gauss–Codazzi–Ricci equations governing the embedding of \mathcal{S} in the spatial hypersurface, and the demand that \mathcal{S} remain at a constant coordinate location. *These equations incorporate no assumption of quasi-equilibrium.*

Terms that vanish because we demand \mathcal{S} be a MOTS, remain a MOTS, or because we demand the horizon to be in equilibrium are in RED.

AH/Quasi-Equilibrium Boundary Conditions

$$\begin{aligned} \theta &= 0 \\ 0 &= \mathcal{D}(\beta_{\perp} - \alpha), \\ \acute{\theta} s^i \bar{\nabla}_i \alpha &= -\frac{1}{\sqrt{2}} \left[\acute{\theta} \left(\acute{\theta} - \frac{1}{\sqrt{2}} K \right) + \acute{\sigma}_{ij} \acute{\sigma}^{ij} \right] (\beta_{\perp} - \alpha) \\ &\quad - \frac{1}{\sqrt{2}} \left[\acute{\theta} \left(\frac{1}{2} \acute{\theta} - \frac{1}{\sqrt{2}} K \right) + \acute{\mathcal{D}} \right] (\beta_{\perp} + \alpha). \end{aligned} \Rightarrow$$

$$\begin{aligned} \tilde{s}^k \tilde{\nabla}_k \ln \psi &= -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J) \\ \beta^i &= \alpha \psi^{-2} \tilde{s}^i + B_{\parallel}^i \\ J \tilde{s}^i \tilde{\nabla}_i \alpha &= -\psi^2 (J^2 - JK + \tilde{\mathcal{D}}) \alpha \end{aligned}$$

$$h_{ij} \equiv \psi^4 \tilde{h}_{ij}$$

$$s^i \equiv \psi^{-2} \tilde{s}^i$$

$$B_{\parallel}^i s_i = 0$$

$$\tilde{\mathcal{D}} \equiv \psi^{-4} [\tilde{h}^{ij} (\check{\nabla}_i - J_i) (\check{\nabla}_j - J_j) - \frac{1}{2} \check{R} + 2 \check{\nabla}^2 \ln \psi]$$

[$\check{\nabla}$ & \check{R} are compatible with \tilde{h}_{ij}]

The conditions of quasi-equilibrium yield boundary conditions for 3 of the 5 constrained variables (ψ , α , β_{\perp}). The remaining two conditions are contained in the definition of β_{\parallel}^i . This freedom is necessary to prescribe the spin of the black hole.

Defining the Spin of the Black Hole

The spin parameters β_{\parallel}^i can be defined by demanding that the MOTS be a *Killing horizon*. The time vector associated with quasi-equilibrium in the corotating frame must be null, forming the null generators of the horizon.

$$k^{\mu} \propto (n^{\mu} + s^{\mu}) \implies k^{\mu} = [1, \alpha s^i - \beta^i]$$

This vector k^{μ} is null for any choice of β^i .

In the frame where a black hole is not spinning, the null time vector has components $t^{\mu} = [1, \vec{0}]$.

Corotating Holes

Corotating holes are at rest in the corotating frame, where we must pose boundary conditions. So,

$$k^{\mu} = [1, \alpha s^i - \beta^i] = [1, \vec{0}]$$

Thus we find

$$\beta^i = \alpha s^i \implies \beta_{\parallel}^i = 0$$

Irrotational Holes

Irrotational holes are at rest in the inertial frame. With the time vectors in the inertial and corotating frames related by

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} + \Omega \frac{\partial}{\partial \phi}$$

$$k^{\mu} = [1, \alpha s^i - \beta^i] = [1, -\Omega(\partial/\partial\phi)^i]$$

Thus we find

$$\beta^i = \alpha s^i + \Omega \left(\frac{\partial}{\partial \phi} \right)^i \implies \beta_{\parallel}^i = \Omega \left(\frac{\partial}{\partial \phi} \right)^i$$

Summary of QE Formalism

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \gamma^{ij} K \quad \tilde{A}^{ij} = \frac{\psi^6}{2\alpha} (\tilde{\mathbb{L}}\beta)^{ij} \quad \partial_t \tilde{\gamma}_{ij} = 0$$

$$\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0$$

$$\tilde{\Delta}_{\mathbb{L}} \beta^i - (\tilde{\mathbb{L}}\beta)^{ij} \tilde{\nabla}_j \ln \alpha \psi^{-6} = \frac{4}{3} \alpha \tilde{\nabla}^i K$$

$$\tilde{\nabla}^2 (\alpha \psi) - (\alpha \psi) \left[\frac{1}{8} \tilde{R} + \frac{5}{12} \psi^4 K^2 + \frac{7}{8} \psi^{-8} \tilde{A}_{ij} \tilde{A}^{ij} \right] = \psi^5 \beta^i \tilde{\nabla}_i K \quad \partial_t K = 0$$

$$\tilde{s}^k \tilde{\nabla}_k \ln \psi|_S = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J)|_S \quad \theta = 0$$

$$\beta^i|_S = \begin{cases} \alpha \psi^{-2} \tilde{s}^i|_S & \text{corotation} \\ \alpha \psi^{-2} \tilde{s}^i|_S + \Omega \tilde{h}^i_j \left(\frac{\partial}{\partial \phi} \right)^j \Big|_S & \text{irrotation} \end{cases} \quad \begin{matrix} \mathcal{L}_\zeta \theta = 0 \\ \sigma_{ij} = 0 \end{matrix}$$

$$J \tilde{s}^i \tilde{\nabla}_i \alpha|_S = -\psi^2 (J^2 - JK + \tilde{\mathcal{D}}) \alpha|_S \quad \mathcal{L}_\zeta \theta = 0$$

$$\begin{aligned} \psi|_{r \rightarrow \infty} &= 1 \\ \beta^i|_{r \rightarrow \infty} &= \Omega \left(\frac{\partial}{\partial \phi} \right)^i \\ \alpha|_{r \rightarrow \infty} &= 1 \end{aligned}$$

The only remaining freedom in the system is the choice of the orbital angular velocity, the initial spatial and temporal gauge, and the initial dynamical (“wave”) content found in Ω , $\tilde{\gamma}_{ij}$ and K .

The Orbital Angular Velocity

- For a given choice of $\tilde{\gamma}_{ij}$ and K , we are still left with a family of solutions parameterized by the orbital angular velocity Ω .
- Except for the case of a single spinning black hole, it is not reasonable to expect more than one value of Ω to correspond to a system in quasi-equilibrium.

GGB[8, 9] have suggested a way to pick the quasi-equilibrium value of Ω :

The Orbital Angular Velocity

- For a given choice of $\tilde{\gamma}_{ij}$ and K , we are still left with a family of solutions parameterized by the orbital angular velocity Ω .
- Except for the case of a single spinning black hole, it is not reasonable to expect more than one value of Ω to correspond to a system in quasi-equilibrium.

GGB[8, 9] have suggested a way to pick the quasi-equilibrium value of Ω :

Ω is chosen as the value for which the ADM mass E_{ADM} equals the Komar mass M_{K} .

Komar mass	$M_{\text{K}} = \frac{1}{4\pi} \oint_{\infty} \gamma^{ij} (\bar{\nabla}_i \alpha - \beta^k K_{ik}) d^2 S_j$	Acceptable definition of the mass <i>only for stationary spacetimes.</i>
------------	---	--

ADM Mass	$E_{\text{ADM}} = \frac{1}{16\pi} \oint_{\infty} \gamma^{ij} \bar{\nabla}_k (\mathcal{G}_i^k - \delta_i^k \mathcal{G}) d^2 S_j$	Acceptable definition of the mass <i>for arbitrary spacetimes.</i> $\mathcal{G}_{ij} \equiv \gamma_{ij} - f_{ij}$
----------	---	--

Do the AH/QE BCs Yield a Well Posed System?

Single Black Hole tests:

- $\tilde{\gamma}_{ij}$ and K from Kerr-Schild:
 - AH/QE BCs seem ill-conditioned with slow/no nonlinear convergence.

Do the AH/QE BCs Yield a Well Posed System?

Single Black Hole tests:

- $\tilde{\gamma}_{ij}$ and K from Kerr-Schild:
 - AH/QE BCs seem ill-conditioned with slow/no nonlinear convergence.
 - Replacing the BC on either α or β_{\perp} with the proper Dirichlet data yields good convergence.

Do the AH/QE BCs Yield a Well Posed System?

Single Black Hole tests:

- $\tilde{\gamma}_{ij}$ and K from Kerr-Schild:
 - AH/QE BCs seem ill-conditioned with slow/no nonlinear convergence.
 - Replacing the BC on either α or β_{\perp} with the proper Dirichlet data yields good convergence.
 - Replacing the BC on either α or β_{\perp} with the **wrong** Dirichlet data yields good convergence.

Do the AH/QE BCs Yield a Well Posed System?

Single Black Hole tests:

- $\tilde{\gamma}_{ij}$ and K from Kerr-Schild:
 - AH/QE BCs seem ill-conditioned with slow/no nonlinear convergence.
 - Replacing the BC on either α or β_{\perp} with the proper Dirichlet data yields good convergence.
 - Replacing the BC on either α or β_{\perp} with the **wrong** Dirichlet data yields good convergence.
 - Solving with Dirichlet BC replacing one of the BCs yields a solution that:
 - ★ obeys the full AH/QE BCs
 - ★ has $\partial_t \psi = 0$
- (if the outer boundary is at ∞)*

Do the AH/QE BCs Yield a Well Posed System?

Single Black Hole tests:

- $\tilde{\gamma}_{ij}$ and K from Kerr-Schild:
 - AH/QE BCs seem ill-conditioned with slow/no nonlinear convergence.
 - Replacing the BC on either α or β_{\perp} with the proper Dirichlet data yields good convergence.
 - Replacing the BC on either α or β_{\perp} with the **wrong** Dirichlet data yields good convergence.
 - Solving with Dirichlet BC replacing one of the BCs yields a solution that:
 - ★ obeys the full AH/QE BCs
 - ★ has $\partial_t \psi = 0$
 - (if the outer boundary is at ∞)*
- $\tilde{\gamma}_{ij} = f_{ij}$ and $K = 1/r^2$ or 0
 - Solving with Dirichlet BC replacing one of the BCs yields a solution that:
 - ★ obeys the full AH/QE BCs
 - ★ has $\partial_t \psi = 0$
 - (if the outer boundary is at ∞)*

References

- [1] T. W. Baumgarte, G. B. Cook, M. A. Scheel, S. L. Shapiro, and S. A. Teukolsky. General relativistic models of binary neutron stars in quasiequilibrium. *Phys. Rev. D*, 57:7299–7311, June 1998. 6
- [2] S. Bonazzola, E. Gourgoulhon, and J.-A. Marck. Numerical models of irrotational binary neutron stars in general relativity. *Phys. Rev. Lett.*, 82:892–895, Feb. 1999. 6
- [3] J. M. Bowen and J. W. York, Jr. Time-asymmetric initial data for black holes and black-hole collisions. *Phys. Rev. D*, 21:2047–2056, Apr. 1980. 3
- [4] S. Brandt and B. Brügmann. A simple construction of initial data for multiple black holes. *Phys. Rev. Lett.*, 78:3606–3609, May 1997. 3
- [5] G. B. Cook. Corotating and irrotational binary black holes in quasi-circular orbit. *Phys. Rev. D*, 65:084003/1–13, Apr. 2002. 0
- [6] G. B. Cook, M. W. Choptuik, M. R. Dubal, S. Klasky, R. A. Matzner, and S. R. Oliveira. Three-dimensional initial data for the collision of two black holes. *Phys. Rev. D*, 47:1471–1490, Feb. 1993. 3
- [7] D. M. Eardley. Black hole boundary conditions and coordinate conditions. *Phys. Rev. D*, 57:2299–2304, Feb. 1998. 9
- [8] E. Gourgoulhon, P. Grandclément, and S. Bonazzola. Binary black holes in circular orbits. I. A global spacetime approach. *Phys. Rev. D*, 65:044020/1–19, Feb. 2002. 6, 14
- [9] P. Grandclément, E. Gourgoulhon, and S. Bonazzola. Binary black holes in circular orbits. II. Numerical methods and first results. *Phys. Rev. D*, 65:044021/1–18, Feb. 2002. 6, 14
- [10] P. Marronetti, G. J. Mathews, and J. R. Wilson. Binary neutron-star systems: From the Newtonian regime to the last stable orbit. *Phys. Rev. D*, 58:107503/1–4, Nov. 1998. 6

- [11] J. Thornburg. Coordinate and boundary conditions for the general relativistic initial data problem. *Class. Quantum Gravit.*, 4:1119–1131, Sept. 1987. 3
- [12] K. Uryū and Y. Eriguchi. New numerical method for constructing quasiequilibrium sequences of irrotational binary neutron stars in general relativity. *Phys. Rev. D*, 61:124023/1–19, June 2000. 6
- [13] J. W. York, Jr. Conformal ‘thin-sandwich’ data for the initial-value problem of general relativity. *Phys. Rev. Lett.*, 82:1350–1353, Feb. 1999. 5