Status of Initial Data for Binary Black Hole Collisions

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Abstract

The first initial data for black-hole binaries were derived from analytic time-symmetric multi-hole solutions of Misner and Lindquist in the early 1960s. These served as a test-bed for all of the pioneering efforts to evolve black-hole binaries to collision. The first major revolution in this field was introduced by Bowen and York in 1980, allowed for time-asymmetric data representing boosted and spinning holes, and required the numerical solution of a single scalar boundary-value problem. Initial-data methods based on the Bowen-York extrinsic curvature were developed and explored over the last 25 years and initial data based on these methods are still widely used for black-hole binary evolutions. However, in the past 5 years, a second major revolution has taken place that promises to yield initial data that is much more astrophysically realistic. These new initial-data sets are more computationally expensive to construct and their full physical content is still being explored. In this talk, we will look at this new method for constructing black-hole binary initial data, see what it does well, and where it needs further improvement.

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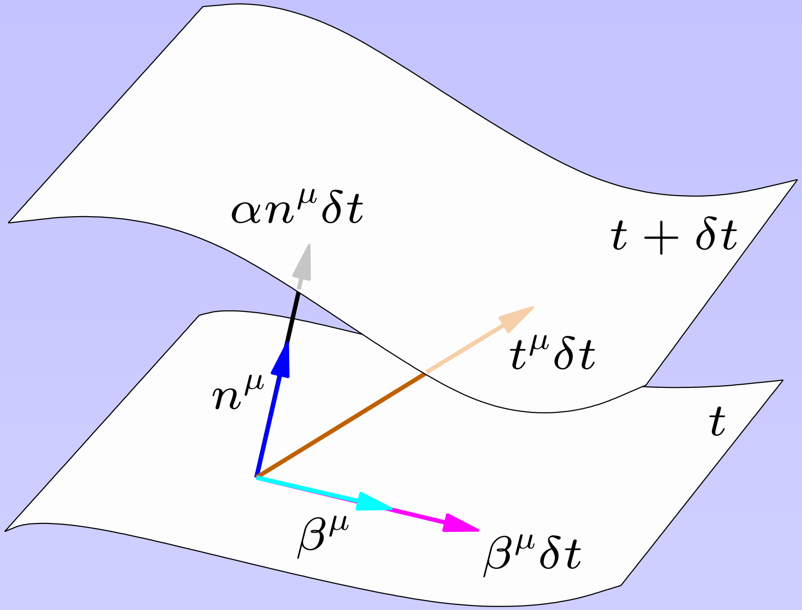
The $3+1$ Decomposition



Lapse : α Spatial metric : γ_{ij}

Shift vector : β^i Extrinsic Curvature : K_{ij}

Time vector : $t^\mu = \alpha n^\mu + \beta^\mu$

$$\mathrm{d}s^2 = -\alpha^2 \mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + \beta^i \mathrm{d}t)(\mathrm{d}x^j + \beta^j \mathrm{d}t)$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \qquad K_{\mu\nu} = -\tfrac{1}{2}\gamma_\mu^\alpha \gamma_\nu^\beta \mathcal{L}_n g_{\alpha\beta}$$

Constraint equations

$$\bar{R} + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

$$\bar{\nabla}_j \left(K^{ij} - \gamma^{ij} K \right) = 8\pi j^i$$

$$S_{\mu\nu} \equiv \gamma_\mu^\alpha \gamma_\nu^\beta T_{\alpha\beta}$$
$$j_\mu \equiv -\gamma_\mu^\nu n^\alpha T_{\nu\alpha}$$
$$\rho \equiv n^\mu n^\nu T_{\mu\nu}$$
$$T_{\mu\nu} = S_{\mu\nu} + 2n_{(\mu} j_{\nu)} + n_\mu n_\nu \rho$$

Evolution equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i$$

$$\begin{aligned} \partial_t K_{ij} = & -\bar{\nabla}_i \bar{\nabla}_j \alpha + \alpha \Big[\bar{R}_{ij} - 2K_{i\ell}K_j^\ell + K K_{ij} \\ & - 8\pi S_{ij} + 4\pi \gamma_{ij}(S - \rho) \Big] \\ & + \beta^\ell \bar{\nabla}_\ell K_{ij} + K_{i\ell} \bar{\nabla}_j \beta^\ell + K_{j\ell} \bar{\nabla}_i \beta^\ell \end{aligned}$$

Degrees of Freedom

Kinematical variables

- Lapse α : 1 degree of freedom
- Shift β^i : 3 degrees of freedom

Initial-data variables

- Metric γ_{ij} : 6 degrees of freedom
- Extrinsic curvature K_{ij} : 6 degrees of freedom

Decomposition of initial-data variables

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \begin{cases} \psi & : \quad \text{1 constrained DOF} \\ \tilde{\gamma}_{ij} & : \quad \left\langle \begin{array}{l} \text{3 spatial gauge DOF} \\ \text{2 dynamical DOF} \end{array} \right. \end{cases} \Bigg\} \textit{ Freely Specifiable}$$

$$K^{ij} = \psi^{-10}\left[\frac{1}{\tilde{\sigma}}(\tilde{\mathbb{L}}V)^{ij} + \tilde{M}^{ij}\right] + \tfrac{1}{3}\psi^{-4}\tilde{\gamma}^{ij}K \begin{cases} V^i & : \left\langle \begin{array}{l} (\tilde{\mathbb{L}}V)^{ij} \equiv \tilde{\nabla}^i V^j + \tilde{\nabla}^i V^j - \tfrac{2}{3}\tilde{\gamma}^{ij}\tilde{\nabla}_k V^k \\ \text{3 constrained DOF} \end{array} \right. \\ \tilde{M}^{ij} & : \left\langle \begin{array}{l} \tilde{\nabla}_j \tilde{M}^{ij} = \tilde{\nabla}_j\left(\frac{1}{\tilde{\sigma}}(\tilde{\mathbb{L}}X)^{ij}\right) \\ \text{2 dynamical DOF} \end{array} \right. \\ K & : \text{1 temporal gauge DOF} \\ \tilde{\sigma} & : \textit{Defn. of TT decomp.} \end{cases} \Bigg\} \textit{ Freely Specifiable}$$

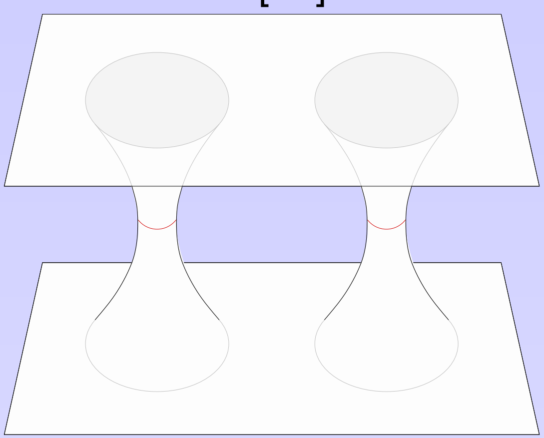
1^{st}-Generation Binary Black-Hole ID
1963

Conformal flatness and time symmetry

$$\left.\begin{aligned}\tilde{\gamma}_{ij} &= f_{ij}\ \textit{(flat)}\\ K_{ij} &= 0\end{aligned}\right\} \Rightarrow \tilde{\nabla}^2\psi = 0$$

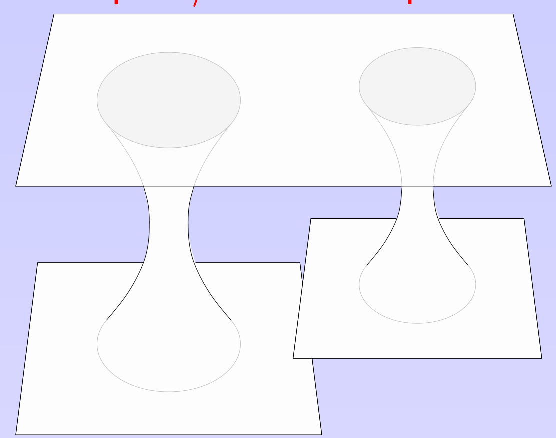
Two general solution schemes: Both yield analytic expressions for ψ.

Misner Data-[16]



Inversion symmetry,
"Two-sheeted"

Lindquist/Brill-Lindquist Data-[14, 5]



"Three-sheeted"

Both methods can model *instantaneously stationary* black holes of various sizes and separations.

2^{nd}-Generation Binary Black-Hole ID

1980's–1990's

Conformal flatness and maximal slicing

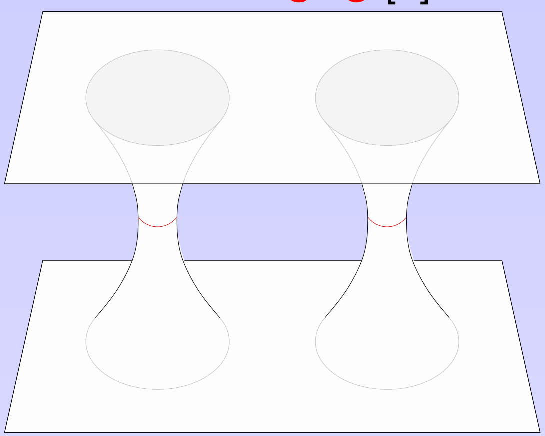
$$\left.\begin{aligned} \tilde{\gamma}_{ij} &= f_{ij}\ \textit{(flat)} \\ \tilde{M}^{ij} &= 0 \\ K &= 0 \\ \tilde{\sigma} &= 1 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \tilde{\nabla}_i(\tilde{\mathbb{L}}V)^{ij} &= 0 \quad \Rightarrow \\ \tilde{\nabla}^2\psi + \tfrac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} &= 0 \quad \Rightarrow \end{aligned}\right.$$

Bowen-York solution[3] 1980
Analytic solutions for $\tilde{A}^{ij}$

Numerical solution for ψ

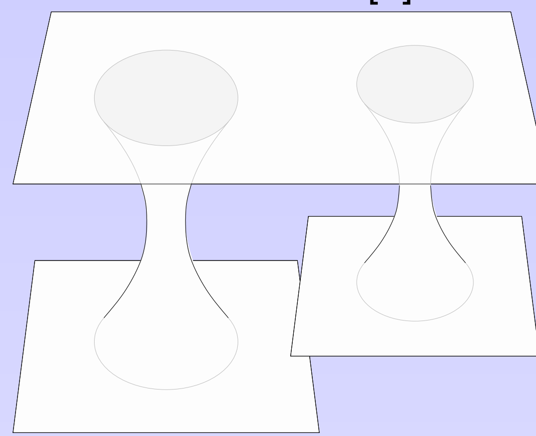
Three general solution schemes

Conformal Imaging-[8] 1990 (2D), 1993 (3D)



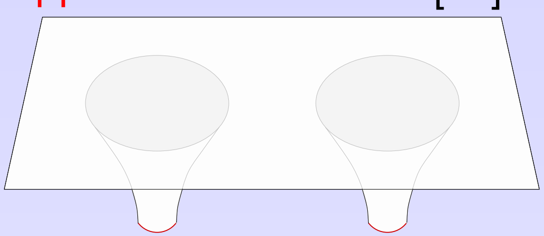
Inversion symmetry inner-BC

Puncture Method-[4] 1997 (3D)



No inner-BC: singular behavior factored out

Apparent Horizon BC-[18] 1987 (2D)



Apparent horizon inner-BC

All methods can produce very general configurations of multiple black holes, but are fundamentally limited by choices for $\tilde{\gamma}_{ij}$ and Bowen-York $\tilde{A}^{ij}$.

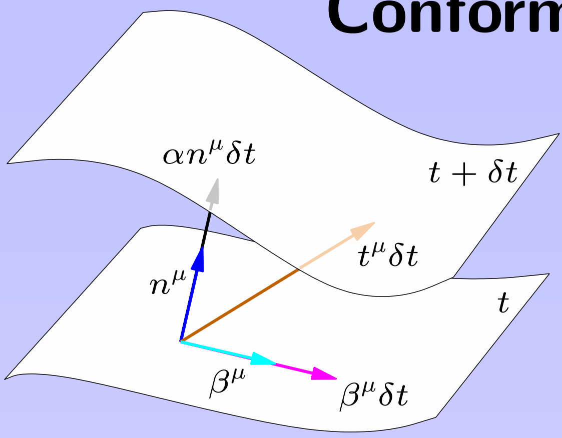
3^{rd}-Generation Binary Black-Hole ID
2000–

Methods that incorporate numerical solution of the momentum constraints but use fixed choices for $\tilde{\gamma}_{ij}$ and $\tilde{M}^{ij}$.

- Kerr background
  - Conformal TT/Boundary Conditions?: 2000[15, 17]
  - Conformal Thin-Sandwich/Boundary Conditions?: 2002[17]
- Helical Killing vector/Quasi-equilibrium
  - Conformal Thin-Sandwich/Inversion Symmetry: 2002[11, 12]
  - ⋆ Conformal Thin-Sandwich/AH: 2002[7, 9]
  - Conformal Thin-Sandwich/Puncture: 2005[13]
  - Conformal Thin-Sandwich/Excision: 2004[21]
  - Conformal TT[1]/Puncture: 2004[19]
- PN background
  - Conformal TT/Puncture: 2003[20]

[1]Uses analytic B-Y vector potential instead of numerically solving momentum constraints, but also constructs lapse and shift through elliptic equations..

Conformal Thin-Sandwich Decomposition



$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$$

$$K^{ij} = \frac{\psi^{-10}}{2\tilde{\alpha}} \left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right] + \tfrac{1}{3}\psi^{-4}\tilde{\gamma}^{ij} K$$

Hamiltonian Const. $\tilde{\nabla}^2\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{12}\psi^5 K^2 + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = -2\pi\psi^5\rho$

Momentum Const. $\tilde{\nabla}_j(\tilde{\mathbb{L}}\beta)^{ij} - (\tilde{\mathbb{L}}\beta)^{ij}\tilde{\nabla}_j\tilde{\alpha} = \frac{4}{3}\tilde{\alpha}\psi^6\tilde{\nabla}^i K + \tilde{\alpha}\tilde{\nabla}_j\left(\frac{1}{\tilde{\alpha}}\tilde{u}^{ij}\right) + 16\pi\tilde{\alpha}\psi^{10}j^i$

Const. Tr(K) eqn.
$$\tilde{\nabla}^2(\psi^7\tilde{\alpha}) - (\psi^7\tilde{\alpha})\left[\tfrac{1}{8}\psi\tilde{R} + \tfrac{5}{12}\psi^5K^2 + \tfrac{7}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} - \psi^5\beta^i\tilde{\nabla}_iK\right] = -2\pi\psi^5K(\rho + 2S) - \psi^5\partial_tK$$

$$\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}}\left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij}\right]$$

Constrained vars : ψ, β^i, and $\tilde{\alpha} \equiv \psi^{-6}\alpha$

Freely specified : $\tilde{\gamma}_{ij}$ $\quad \tilde{u}^{ij} \equiv -\partial_t\tilde{\gamma}^{ij}$

K *and* $\partial_t K$

Quasiequilibrium $\Rightarrow \begin{cases} \partial_t\tilde{\gamma}^{ij} = 0 \\ \partial_t K = 0 \end{cases}$

AH and QE Conditions on the Inner Boundary

The quasiequilibrium inner boundary conditions start with the following assumptions:

1. The inner boundary $\mathcal{S}$ is a (MOTS):
   marginally outer-trapped surface

   $\rightarrow \quad \theta = 0$ (expansion)

2. The horizons are in quasiequilibrium:

   $\rightarrow \quad \sigma_{ij} = 0$ and no matter is on $\mathcal{S}$ (shear)

   Raychaudhuri's equation implies that MOTS initially evolves along k^μ.

   $$\mathcal{L}_k\theta = \tfrac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} - \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}k^\mu k^\nu = 0$$

3. The time evolution vector lies in outgoing null surface through $\mathcal{S}$:

   $\rightarrow \quad t^\mu k_\mu|_{\mathcal{S}} = 0$

$$\begin{aligned}
\tilde{s}^k\tilde{\nabla}_k \ln\psi &= -\tfrac{1}{4}(\tilde{h}^{ij}\tilde{\nabla}_i\tilde{s}_j - \psi^2 J) \\
\beta^i &= \alpha\psi^{-2}\tilde{s}^i + {\beta_\parallel}^i \\
0 &= \tilde{D}_{(i}\beta_{\parallel j)} - \tfrac{1}{2}\tilde{h}_{ij}\tilde{D}_k{\beta_\parallel}^k
\end{aligned}$$

Fixing the Orbital Angular Velocity

Komar-mass ansatz:

Ω_0 is chosen as the value for which the ADM energy E_{ADM} equals the Komar mass M_{K}.

Komar mass

$$M_{\mathrm{K}} = \frac{1}{4\pi}\oint_\infty \gamma^{ij}(\bar{\nabla}_i\alpha - \beta^k K_{ik})\mathrm{d}^2 S_j$$

Acceptable definition of the mass *only for stationary spacetimes*.

ADM energy

$$E_{\mathrm{ADM}} = \frac{1}{16\pi}\oint_\infty \gamma^{ij}\bar{\nabla}_k(\mathcal{G}_i^k - \delta_i^k\mathcal{G})\mathrm{d}^2 S_j$$

Acceptable definition of the mass *for arbitrary spacetimes*.

$\mathcal{G}_{ij} \equiv \gamma_{ij} - f_{ij}$

Effective Potential Method:

Ω_0 is chosen so that the ADM energy E_{ADM} is a minimum for a sequence of models along which the total angular momentum remains fixed.

Measuring the Spin of a Black Hole

- Spin is only rigorously defined at spatial/null infinity.
- Must use *quasi-local* definition: e.g. Brown & York[6] or Ashtekar & Krishnan[2]

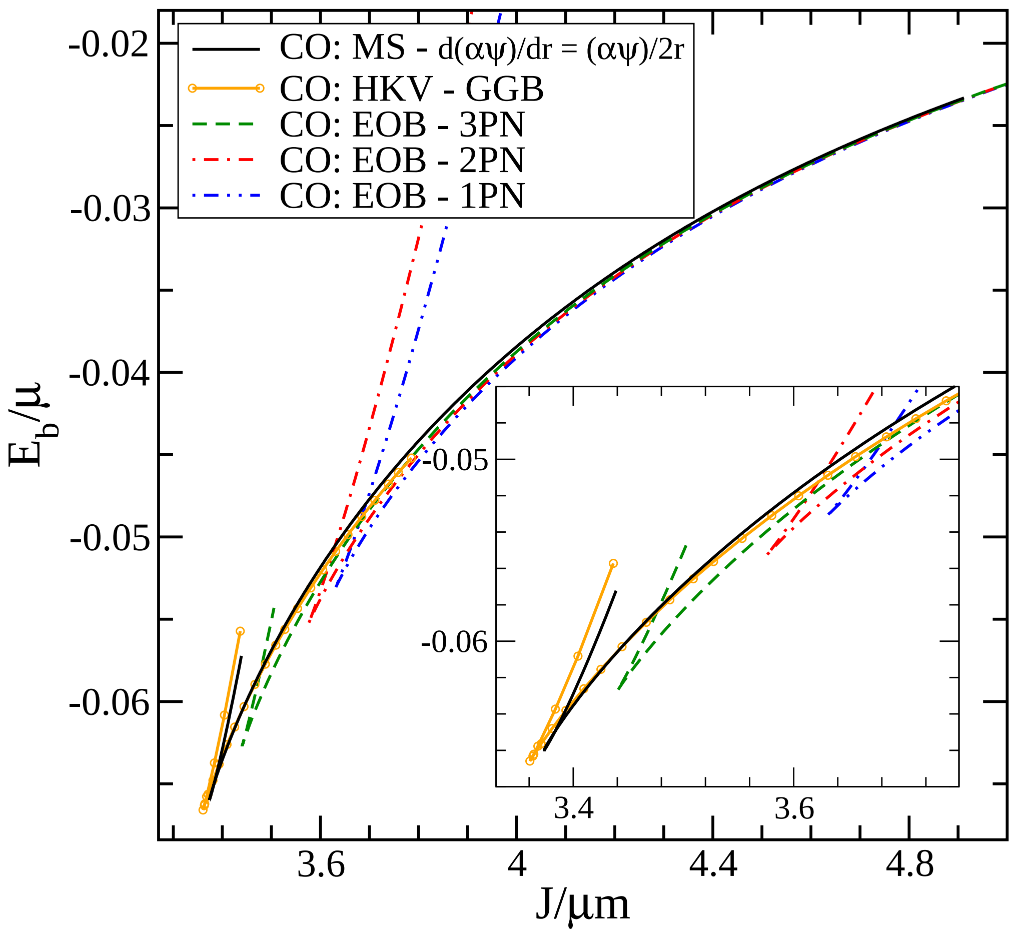
$$
\begin{aligned}
S &= \frac{1}{8\pi} \oint_{BH} K_{ij} \xi^i s^j \sqrt{h} d^2x \\
&= \frac{1}{8\pi} \oint_{BH} \tilde{A}_{ij} \xi^i \tilde{s}^j \sqrt{\tilde{h}} d^2x
\end{aligned}
$$

$\xi^i \tilde{s}_i = 0$

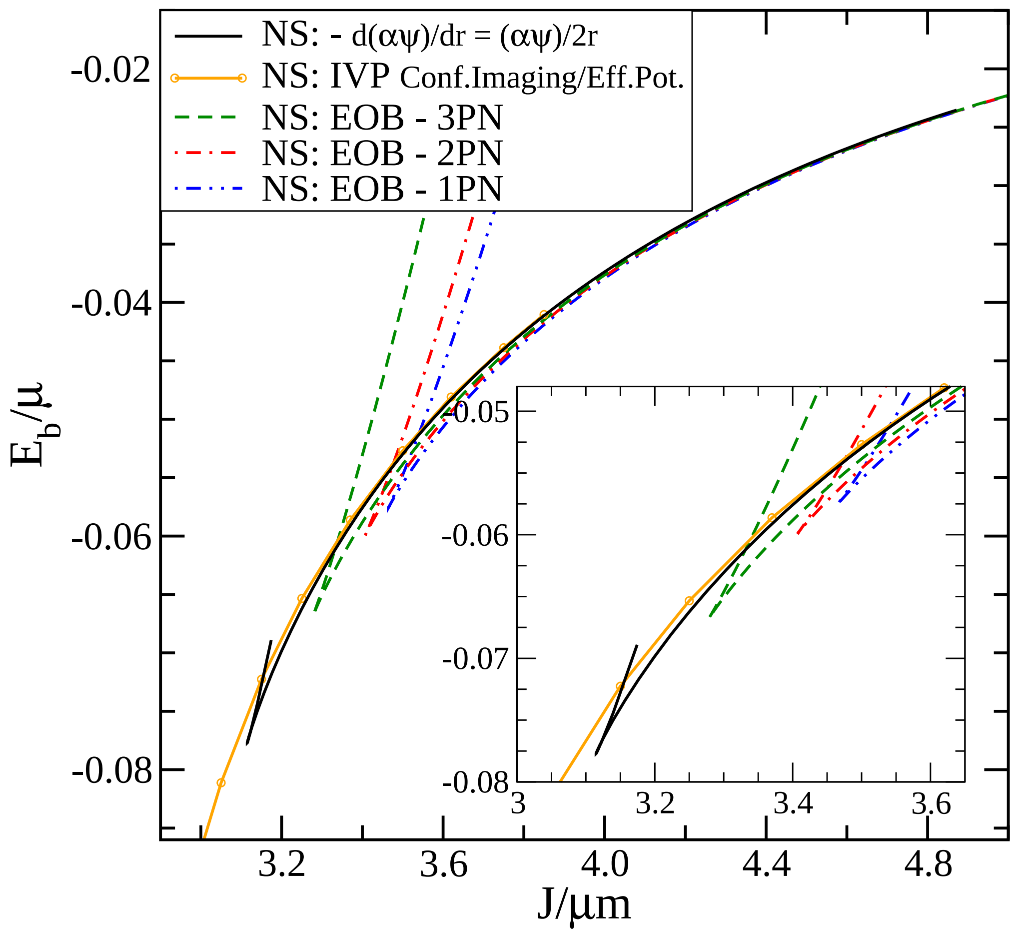
$$
\xi^i = \begin{cases} \xi^i_{\mathsf{CK}} & : \quad \text{Killing vector of } \tilde{h}_{ij} \Rightarrow \text{conformal Killing vector of } h_{ij} \\ \xi^i_{\mathsf{KV}} & : \quad \text{Killing vector of } h_{ij} \ \textit{(Approximate)} \end{cases}
$$

Equal Mass; Maximal Slice; E_b/μ vs $J/\mu m$

Corotation



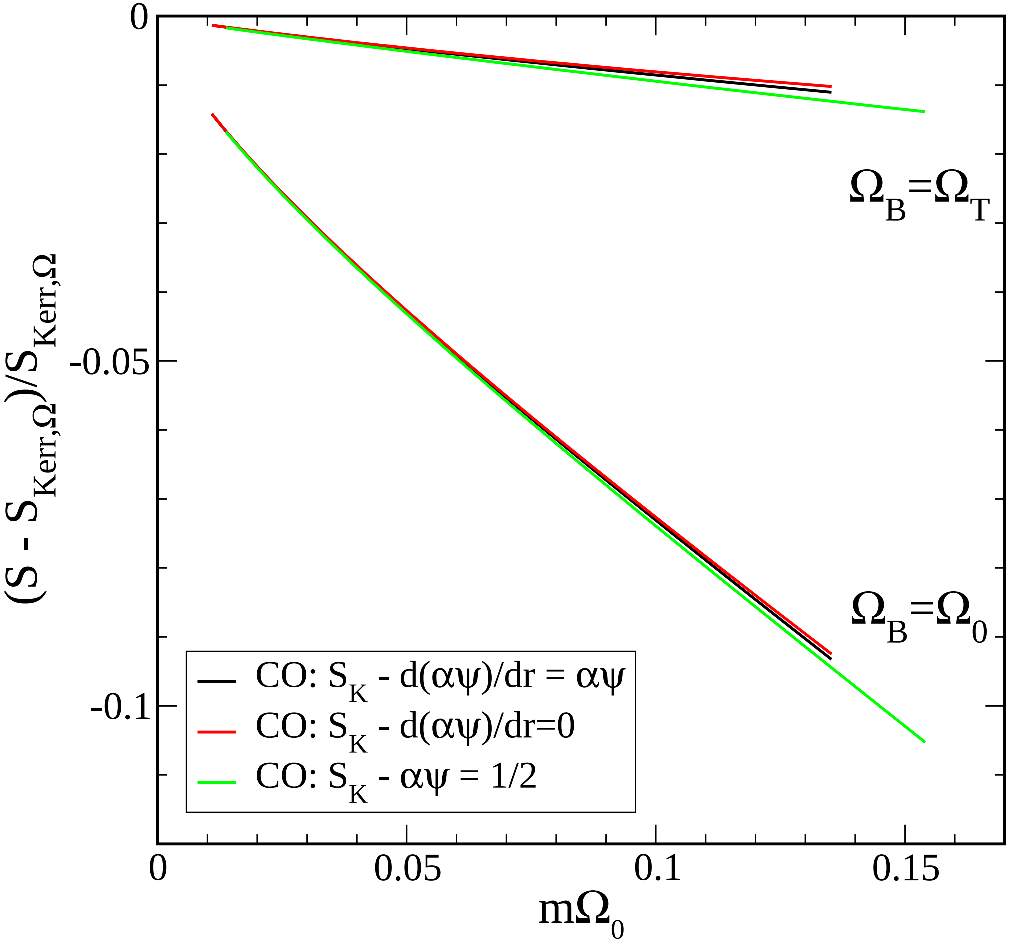
Non-Spinning

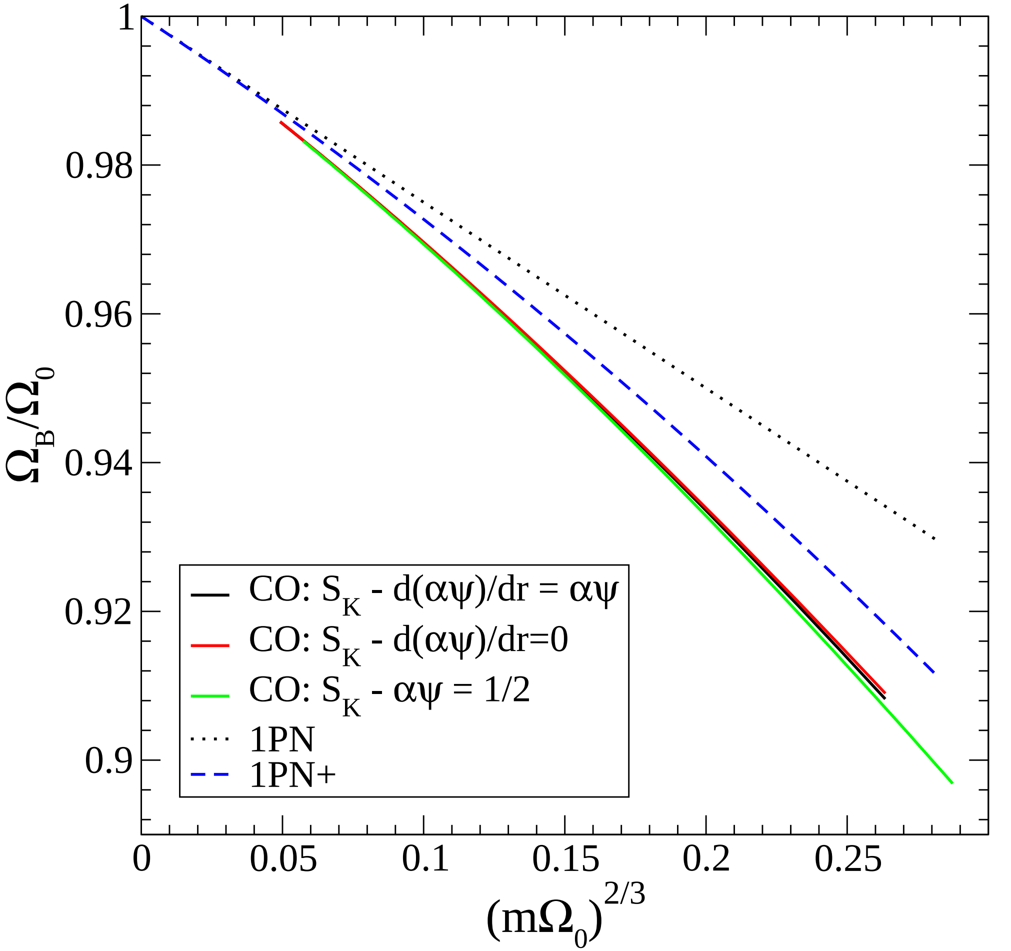


Corotation Spin

$$S_{\rm Kerr}(M_{\rm irr},\Omega_{\rm B}) = \frac{4M_{\rm irr}^3\Omega_{\rm B}}{\sqrt{1-4(M_{\rm irr}\Omega_{\rm B})^2}}$$

$$\frac{\Omega_{\rm B}}{\Omega_0} = \frac{1}{m\Omega_0}\frac{S/M_{\rm irr}^2}{\sqrt{4+\left(S/M_{\rm irr}^2\right)^2}}$$



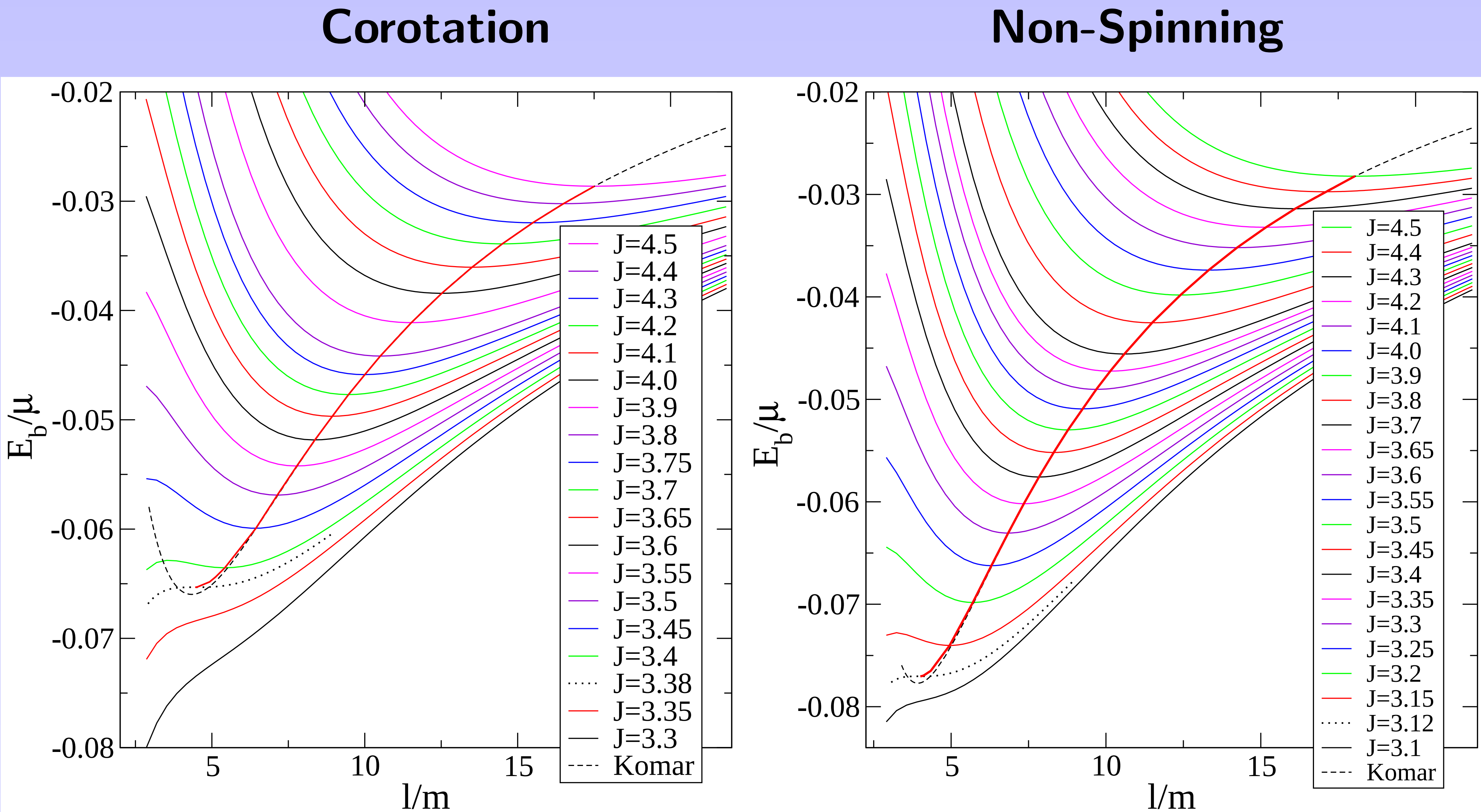


$$\Omega_{\rm T} = \Omega_0\left[1-\eta\left(\frac{m}{b}\right)+O\left(\frac{m}{b}\right)^{3/2}\right]$$ Alvi[1]

$$\frac{m}{b} = (m\Omega_0)^{2/3}\left[1+\left(1-\frac{1}{3}\eta\right)(m\Omega_0)^{2/3}+O\,(m\Omega_0)^{4/3}\right]$$

$$\frac{\Omega_{\rm T}}{\Omega_0} = 1-\eta\,(m\Omega_0)^{2/3} - \eta\left(1-\frac{1}{3}\eta\right)(m\Omega_0)^{4/3}+\ldots$$

Equal Mass; Effective Potential; E_b/μ vs ℓ/m

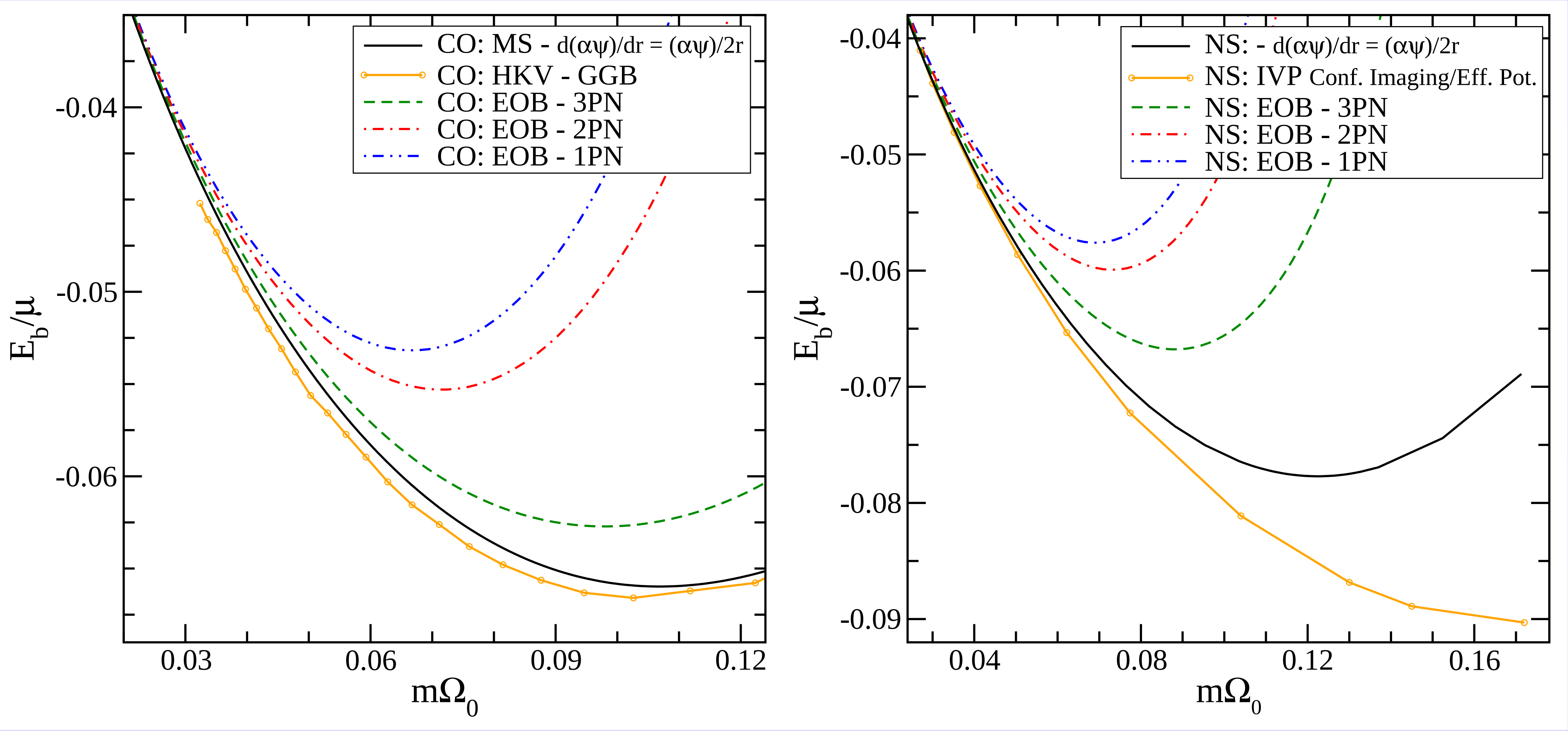


$$\delta E_{\text{ADM}} = \Omega_0 \delta J_{\text{ADM}} + \sum \kappa_i \delta \mathcal{A}_i [10]$$

Equal Mass; Maximal Slice; E_b/μ vs $m\Omega$

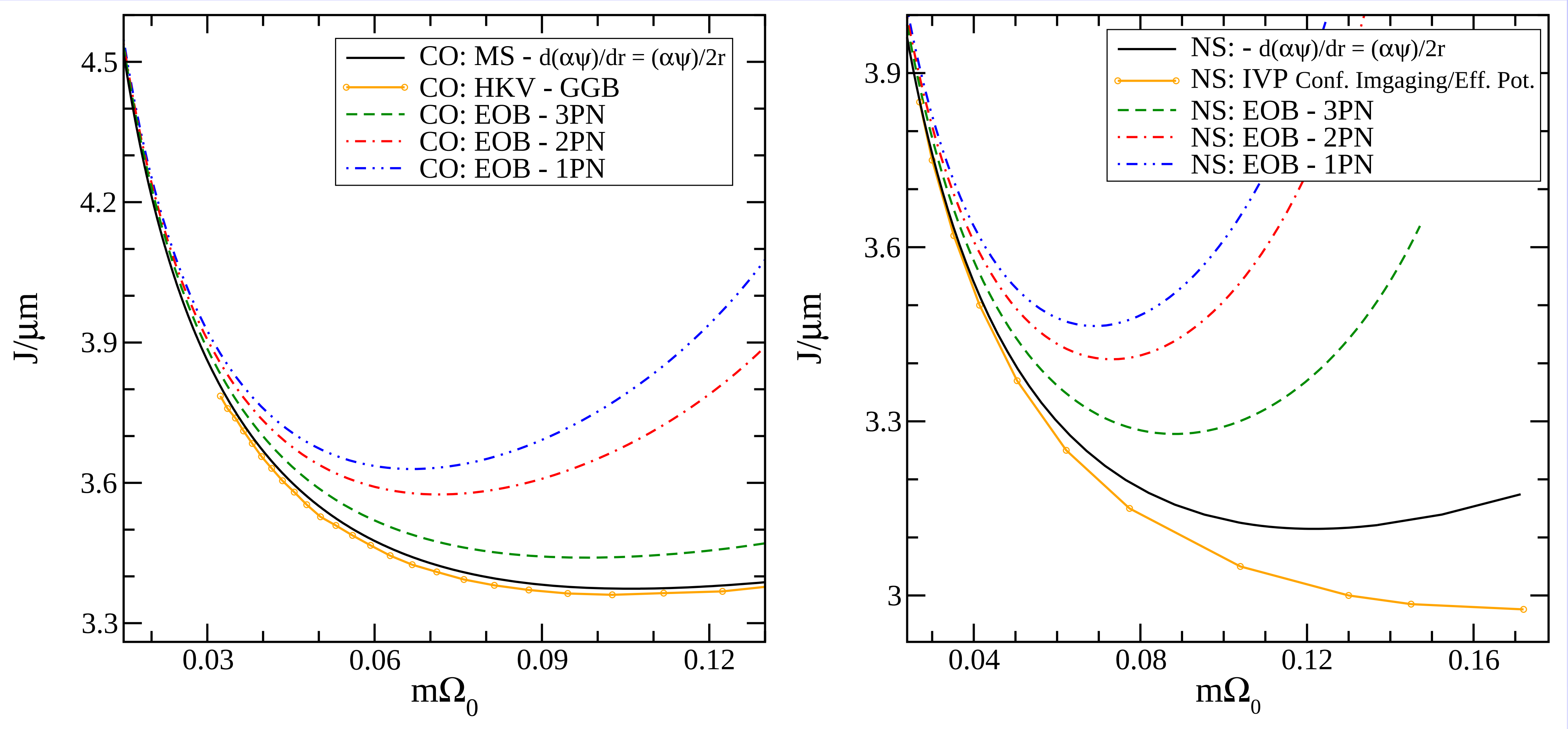
Corotation

Non-Spinning



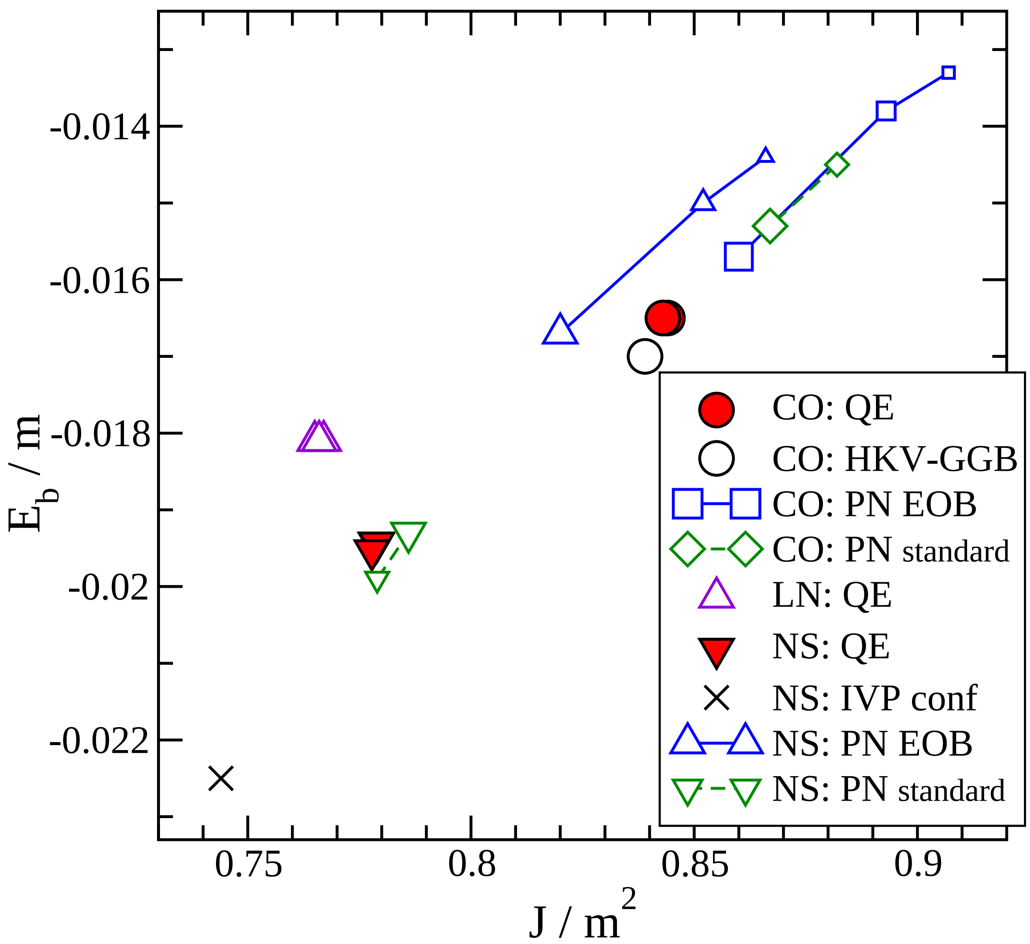
Equal Mass; Maximal Slice; $J/\mu m$ vs $m\Omega$

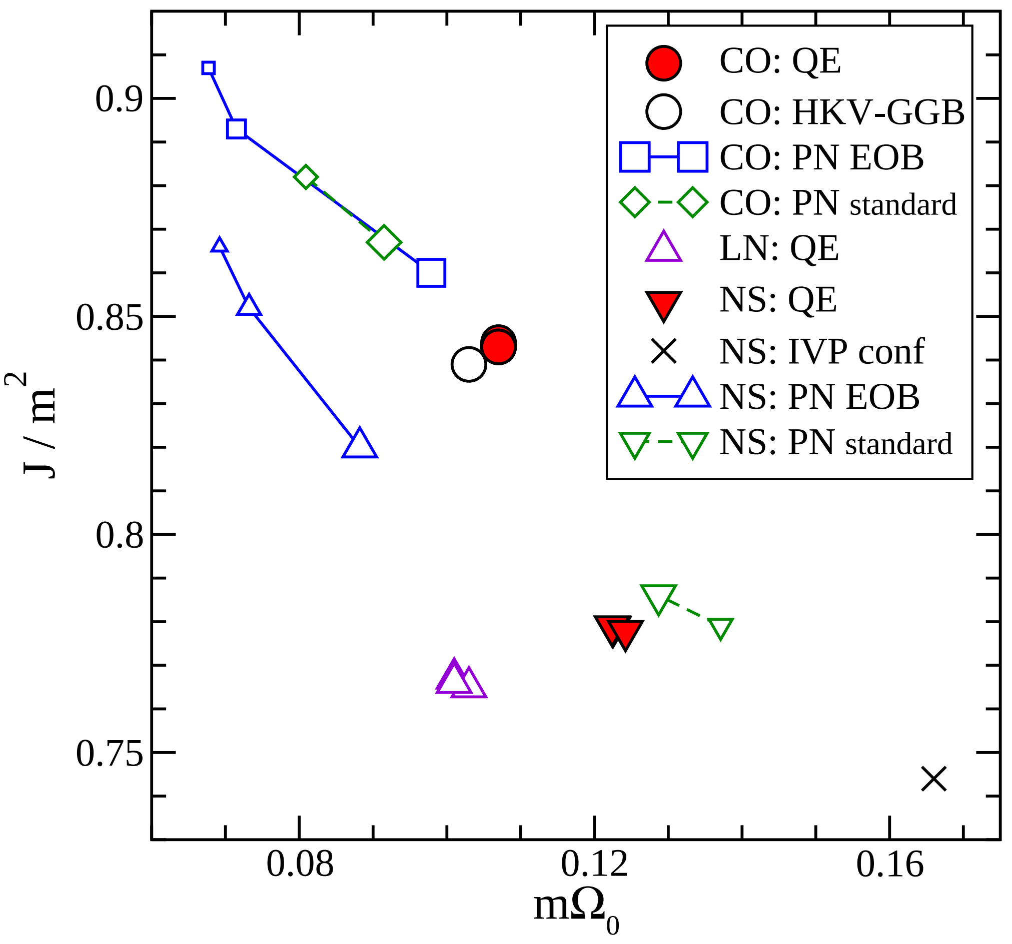
Corotation　　　　Non-Spinning



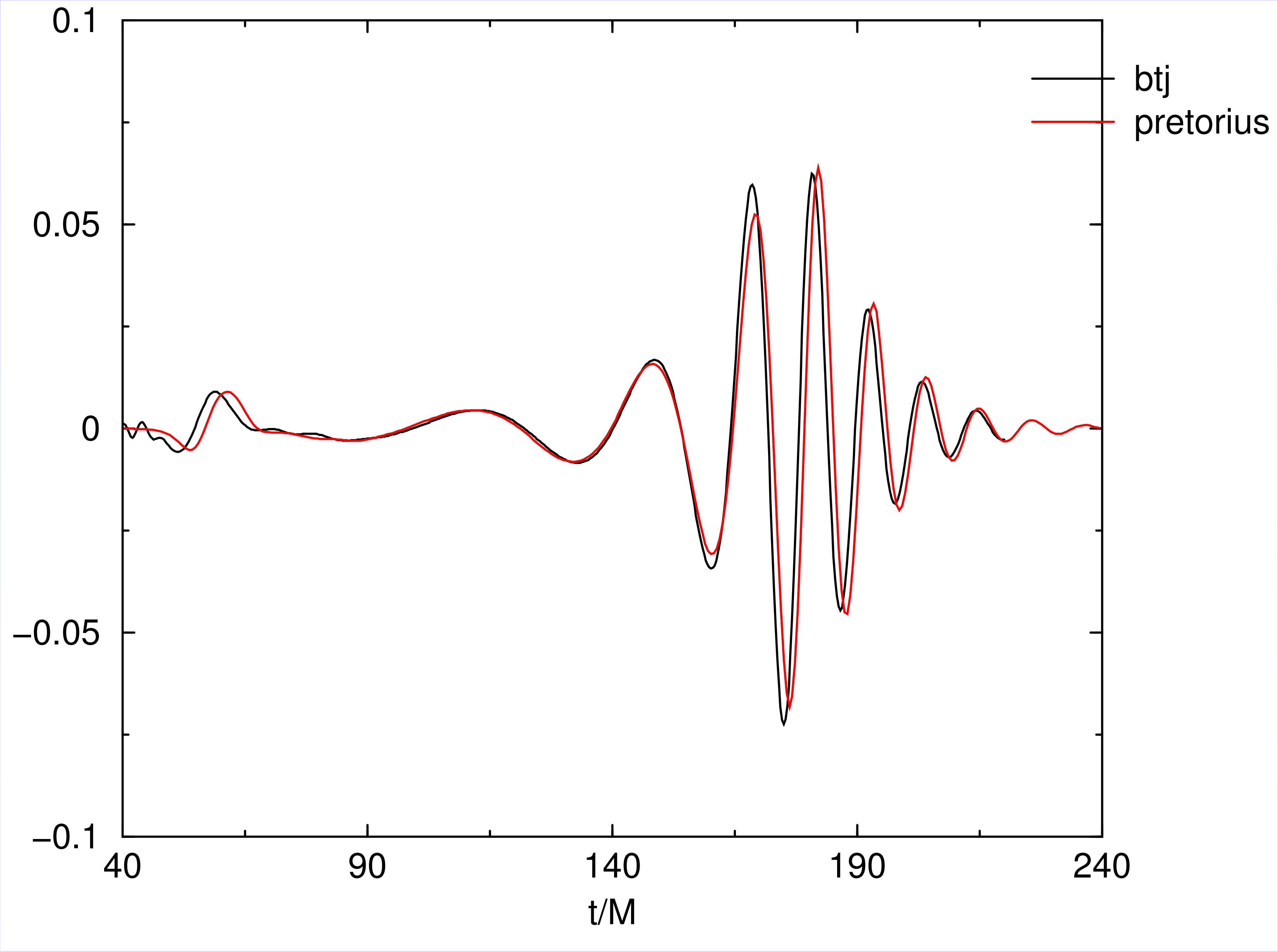
Equal Mass; Comparison of ISCO

E_b/m vs J/m^2 $\qquad$ J/m vs $m\Omega$

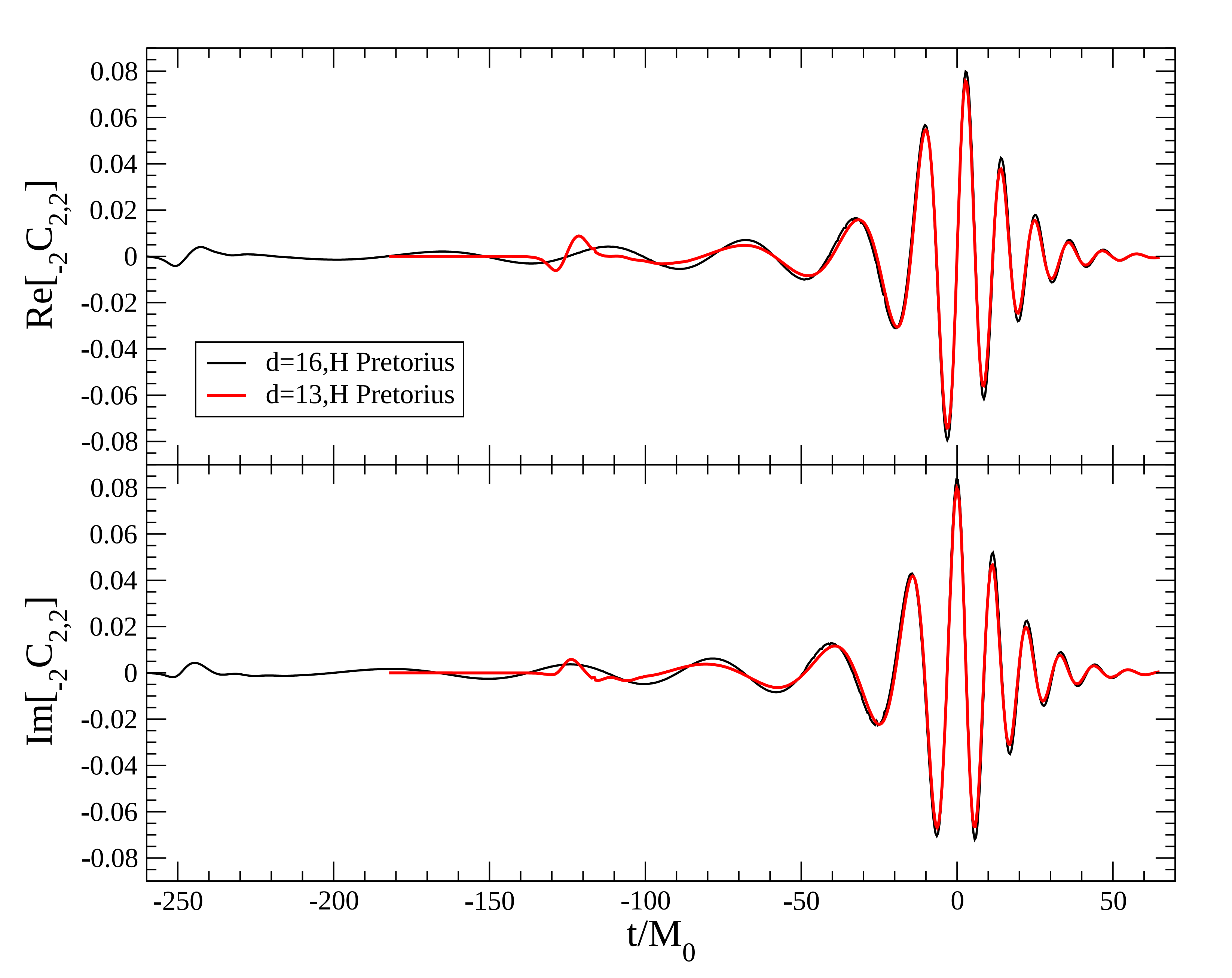




Ψ_4 waveforms from 2^{nd} and 3^{rd} Generation Initial Data.



Ψ_4 waveforms from $d = 16$ & $d = 13$ corotating ID.



Open Questions

- Arbitrary Spins.

- Unequal masses.

- Can we use this approach to construct binaries in elliptical orbits?

* Can we incorporate a *non-flat* conformal metric that is consistent with quasiequilibrium?
  - Will this reduce the initial wave transient?
  - Can we build in the initial outgoing wave?

* Can we build in an appropriate initial radial velocity?

References

[1] K. Alvi. Approximte binary-black-hole metric. *Phys. Rev. D*, 61:124013/1–19, June 2000. 11

[2] A. Ashtekar and B. Krishnan. Dynamical horizons and their properties. *Phys. Rev. D*, 68:104030/1–25, 2003. 9

[3] J. M. Bowen and J. W. York, Jr. Time-asymmetric initial data for black holes and black-hole collisions. *Phys. Rev. D*, 21:2047–2056, Apr. 1980. 4

[4] S. Brandt and B. Brügmann. A simple construction of initial data for multiple black holes. *Phys. Rev. Lett.*, 78:3606–3609, May 1997. 4

[5] D. R. Brill and R. W. Lindquist. Interaction energy in geometrostatics. *Phys. Rev.*, 131:471–476, July 1963. 3

[6] J. D. Brown and J. W. York, Jr. Quasilocal energy and conserved charges derived from the gravitational action. *Phys. Rev. D*, 47:1407–1419, 1993. 9

[7] G. B. Cook. Corotating and irrotational binary black holes in quasi-circular orbit. *Phys. Rev. D*, 65:084003/1–13, Apr. 2002. 5

[8] G. B. Cook, M. W. Choptuik, M. R. Dubal, S. Klasky, R. A. Matzner, and S. R. Oliveira. Three-dimensional initial data for the collision of two black holes. *Phys. Rev. D*, 47:1471–1490, Feb. 1993. 4

[9] G. B. Cook and H. P. Pfeiffer. Excision boundary conditions for black hole initial data. *Phys. Rev. D*, 70:104016/1–24, 2004. 5

[10] J. L. Friedman, K. Uryū, and M. Shibata. Thermodynamics of binary black holes and neutron stars. *Phys. Rev. D*, 65:064035/1–20, Mar. 2002. 12

[11] E. Gourgoulhon, P. Grandclément, and S. Bonazzola. Binary black holes in circular orbits. I. A global spacetime approach. *Phys. Rev. D*, 65:044020/1–19, Feb. 2002. 5

[12] P. Grandclément, E. Gourgoulhon, and S. Bonazzola. Binary black holes in circular orbits. II. Numerical methods and first results. *Phys. Rev. D*, 65:044021/1–18, Feb. 2002. 5

[13] M. D. Hannam. Quasi-circular orbits for conformal thin-sandwich puncture binary black holes. *Phys. Rev. D*, 72:044025/1–8, Aug. 2005. 5

[14] R. W. Lindquist. Initial-value problem on Einstein-Rosen manifolds. *J. Math. Phys.*, 4:938–950, July 1963. 3

[15] P. Marronetti and R. A. Matzner. Solving the initial value problem of two black holes. *Phys. Rev. Lett.*, 85:5500–5503, Dec. 2000. 5

[16] C. W. Misner. The method of images in geometrostatics. *Ann. Phys.*, 24:102–117, Oct. 1963. 3

[17] H. P. Pfeiffer, G. B. Cook, and S. A. Teukolsky. Comparing initial-data sets for binary black holes. *Phys. Rev. D*, 66:024047/1–17, July 2002. 5

[18] J. Thornburg. Coordinate and boundary conditions for the general relativistic initial data problem. *Class. Quantum Gravit.*, 4:1119–1131, Sept. 1987. 4

[19] W. Tichy and B. Brügmann. Quasiequilibrium binary black hole sequences for puncture data derived from helical Killing vector conditions. *Phys. Rev. D*, 69:024006/1–7, Jan. 2004. 5

[20] W. Tichy, B. Brügmann, M. Campanelli, and P. Diener. Binary black hole initial data for numerical general relativity based on post-newtonian data. *Phys. Rev. D*, 67:064008/1–13, Mar. 2003. 5

[21] H.-J. Yo, J. N. Cook, S. L. Shapiro, and T. W. Baumgarte. Quasi-equilibrium binary black hole initial data for dynamical evolutions. *Phys. Rev. D*, 70:084033/1–14, Oct. 2004. 5