

Initial Data for Black-Hole Binaries

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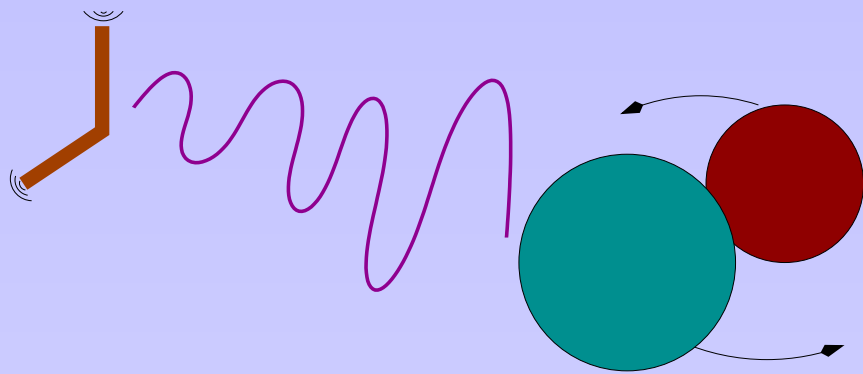
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Abstract

We will examine the current state of our efforts to generate astrophysically realistic initial data for black-hole binaries.

Collaborators: Harald Pfeiffer (Caltech) & Saul Teukolsky (Cornell)

Motivation

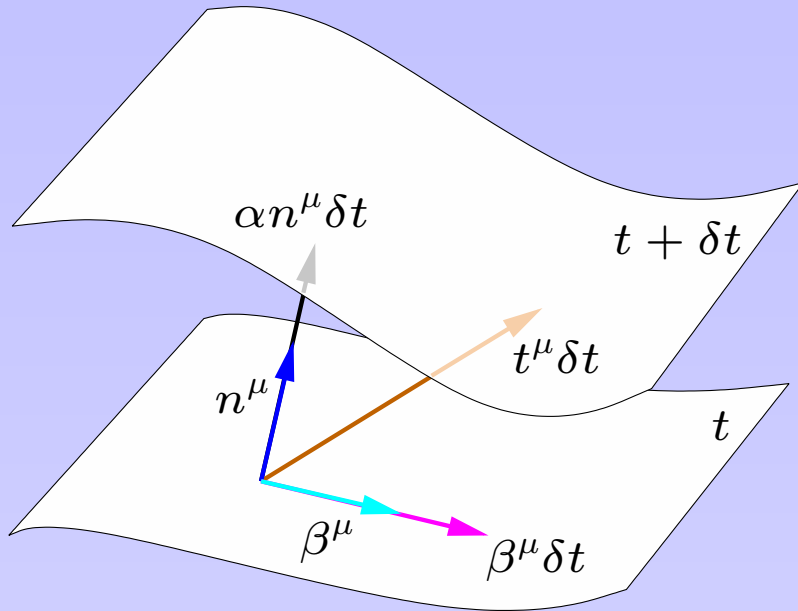


- Black hole binaries are among the most likely sources for early detection with LIGO, VIRGO, GEO, . . .
- Available computed waveforms should increase chance of detecting collision events.

Quasi-Equilibrium Binary Data

- General Relativity doesn't permit *true* equilibrium for astrophysical binary systems.
- When the bodies are sufficiently far apart, the timescale for orbital decay is much larger than the orbital period.
- If the orbit is nearly circular (*quasi-circular*) then there is a *corotating reference frame* in which the binary appears to be at rest.
- ★ Quasi-equilibrium gives us a *physical condition* to guide us in fixing boundary conditions and data that is not otherwise constrained.

The 3 + 1 Decomposition



Lapse : α

Spatial metric : γ_{ij}

Shift vector : β^i

Extrinsic Curvature : K_{ij}

Time vector : $t^\mu = \alpha n^\mu + \beta^\mu$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \quad K_{\mu\nu} = -\frac{1}{2}\gamma_\mu^\alpha \gamma_\nu^\beta \mathcal{L}_n g_{\alpha\beta}$$

Constraint equations

$$\bar{R} + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

$$\bar{\nabla}_j (K^{ij} - \gamma^{ij}K) = 8\pi j^i$$

$$S_{\mu\nu} \equiv \gamma_\mu^\alpha \gamma_\nu^\beta T_{\alpha\beta}$$

$$j_\mu \equiv -\gamma_\mu^\nu n^\alpha T_{\nu\alpha}$$

$$\rho \equiv n^\mu n^\nu T_{\mu\nu}$$

$$T_{\mu\nu} = S_{\mu\nu} + 2n_{(\mu}j_{\nu)} + n_\mu n_\nu \rho$$

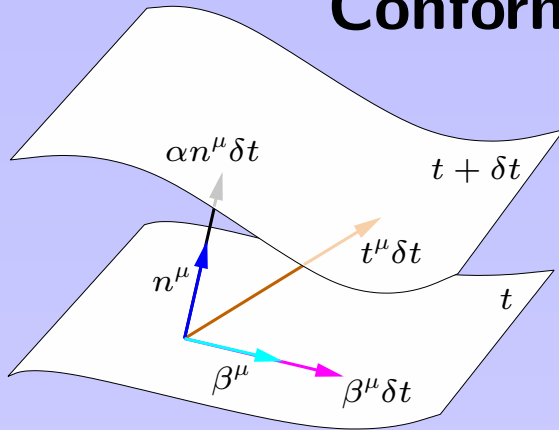
Evolution equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i$$

$$\partial_t K_{ij} = -\bar{\nabla}_i \bar{\nabla}_j \alpha + \alpha \left[\bar{R}_{ij} - 2K_{il}K_j^\ell + K K_{ij} - 8\pi S_{ij} + 4\pi \gamma_{ij}(S - \rho) \right]$$

$$+ \beta^\ell \bar{\nabla}_\ell K_{ij} + K_{il} \bar{\nabla}_j \beta^\ell + K_{jl} \bar{\nabla}_i \beta^\ell$$

Conformal Thin-Sandwich Decomposition



$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$$

$$K^{ij} = \frac{\psi^{-10}}{2\tilde{\alpha}} \left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right] + \frac{1}{3}\psi^{-4}\tilde{\gamma}^{ij}K$$

Hamiltonian Const. $\tilde{\nabla}^2\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{12}\psi^5K^2 + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = -2\pi\psi^5\rho$

Momentum Const. $\tilde{\nabla}_j(\tilde{\mathbb{L}}\beta)^{ij} - (\tilde{\mathbb{L}}\beta)^{ij}\tilde{\nabla}_j\tilde{\alpha} = \frac{4}{3}\tilde{\alpha}\psi^6\tilde{\nabla}^iK + \tilde{\alpha}\tilde{\nabla}_j\left(\frac{1}{\tilde{\alpha}}\tilde{u}^{ij}\right) + 16\pi\tilde{\alpha}\psi^{10}j^i$

Const. Tr(K) eqn. $\tilde{\nabla}^2(\psi^7\tilde{\alpha}) - (\psi^7\tilde{\alpha})\left[\frac{1}{8}\psi\tilde{R} + \frac{5}{12}\psi^5K^2 + \frac{7}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} - \psi^5\beta^i\tilde{\nabla}_iK\right]$
 $= -2\pi\psi^5K(\rho + 2S) - \psi^5\partial_tK$

$$\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right]$$

Constrained vars : ψ , β^i , and $\tilde{\alpha} \equiv \psi^{-6}\alpha$

Freely specified : $\tilde{\gamma}_{ij}$ $\tilde{u}^{ij} \equiv -\partial_t\tilde{\gamma}^{ij}$
 K *and* ∂_tK

$$\text{Quasi-equilibrium} \Rightarrow \begin{cases} \partial_t\tilde{\gamma}^{ij} = 0 \\ \partial_tK = 0 \end{cases}$$

Equations of Quasi-Equilibrium

$$\left. \begin{array}{l} \text{Ham. \& Mom. const.} \\ \text{eqns., \& Const Tr}(K) \\ \text{eqn. from Conf. TS} \\ + \tilde{u}^{ij} = \partial_t K = 0 \end{array} \right\} \Rightarrow \text{Eqns. of Quasi-Equilibrium}$$

With $\tilde{\gamma}_{ij} = f_{ij}$ and $K = 0$, these equations have been widely used to construct binary neutron star initial data [2, 9, 3, 10].

Binary neutron star initial data require:

- boundary conditions at infinity compatible with asymptotic flatness and corotation.

$$\psi|_{r \rightarrow \infty} = 1 \quad \beta^i|_{r \rightarrow \infty} = \Omega \left(\frac{\partial}{\partial \phi} \right)^i \quad \alpha|_{r \rightarrow \infty} = 1$$

- compatible solution of the equations of hydrostatic equilibrium. ($\Rightarrow \Omega$)

Binary black hole initial data require:

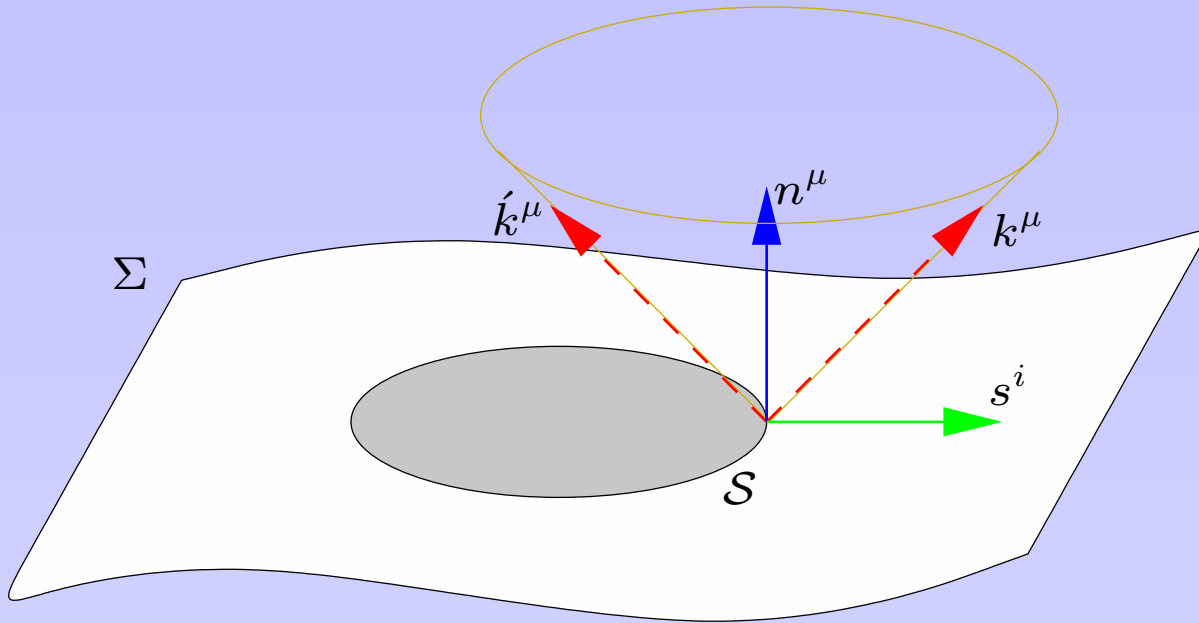
- a means for choosing the angular velocity of the orbit Ω .

★ *with excision*, inner boundary conditions are needed for ψ , β^i , and $\tilde{\alpha}$.

Gourgoulhon, Grandclément, & Bonazzola [7, 8]: Black-hole binaries with $\tilde{\gamma}_{ij} = f_{ij}$ & $K = 0$, “inversion-symmetry”, and “Killing-horizon” conditions on the excision boundaries.

“Solutions” require *constraint violating* regularity condition imposed on inner boundaries!

The Inner Boundary



$$s_i \equiv \frac{\bar{\nabla}_i \tau}{|\bar{\nabla} \tau|}$$

$$h_{ij} \equiv \gamma_{ij} - s_i s_j$$

$$k^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu + s^\mu)$$

$$\hat{k}^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu - s^\mu)$$

Extrinsic curvature of S embedded in spacetime

$$\Sigma_{\mu\nu} \equiv \frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_k g_{\alpha\beta}$$

$$\hat{\Sigma}_{\mu\nu} \equiv \frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_{\hat{k}} g_{\alpha\beta}$$

Extrinsic curvature of S embedded in Σ

$$H_{ij} \equiv \frac{1}{2} h_i^k h_j^\ell \mathcal{L}_s \gamma_{kl}$$

$$\Sigma_{ij} = \frac{1}{\sqrt{2}} (H_{ij} - J_{ij})$$

$$\hat{\Sigma}_{ij} = -\frac{1}{\sqrt{2}} (H_{ij} + J_{ij})$$

Projections of K_{ij} onto S

$$J_{ij} \equiv h_i^k h_j^\ell K_{kl}$$

$$J_i \equiv h_i^k s^\ell K_{kl}$$

$$J \equiv h^{ij} J_{ij} = h^{ij} K_{ij}$$

Expansion of null rays

$$\theta \equiv h^{ij} \Sigma_{ij} = \frac{1}{\sqrt{2}} (H - J)$$

$$\hat{\theta} \equiv h^{ij} \hat{\Sigma}_{ij} = -\frac{1}{\sqrt{2}} (H + J)$$

Shear of null rays

$$\sigma_{ij} \equiv \Sigma_{ij} - \frac{1}{2} h_{ij} \theta$$

$$\hat{\sigma}_{ij} \equiv \hat{\Sigma}_{ij} - \frac{1}{2} h_{ij} \hat{\theta}$$

AH and QE Conditions on the Inner Boundary

The quasi-equilibrium inner boundary conditions start with the following assumptions:

1. The inner boundary \mathcal{S} is a (MOTS):
marginally outer-trapped surface

$$\rightarrow \theta = 0$$

2. The inner boundary \mathcal{S} remains a MOTS:

$$\rightarrow \mathcal{L}_\zeta \theta = 0$$

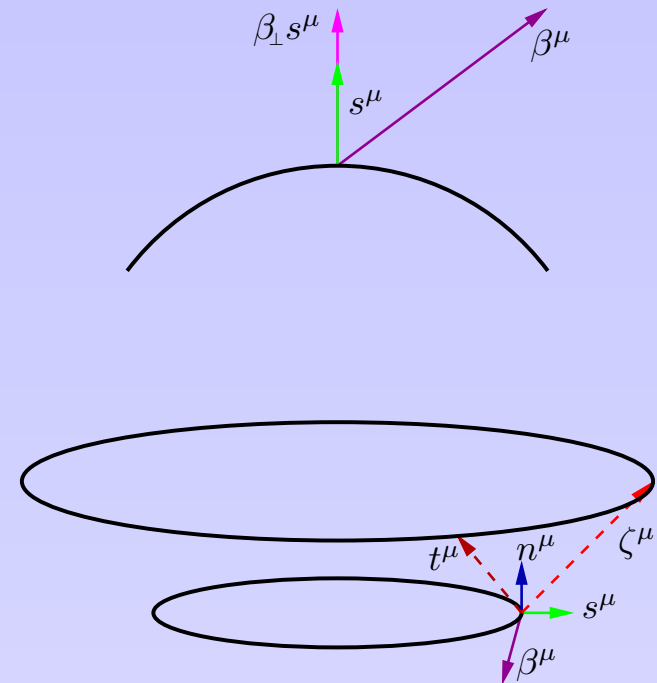
$$t^\mu = \alpha n^\mu + \beta^\mu \quad \zeta^\mu \equiv \alpha n^\mu + \beta_\perp s^\mu$$

$$\beta_\perp \equiv \beta^i s_i$$

ζ^μ is *null* on the AH and the chosen form is a gauge choice.

3. The horizons are in quasi-equilibrium:

$$\rightarrow \sigma_{ij} = 0 \text{ and no matter is on } \mathcal{S}$$



AH/Quasi-Equilibrium Boundary Conditions

$$\theta = \frac{\psi^{-2}}{\sqrt{2}} \left[\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j + 4\tilde{s}^k \tilde{\nabla}_k \ln \psi - \psi^2 J \right]$$

$$\mathcal{L}_\zeta \theta = -\frac{1}{\sqrt{2}} \left[\theta(\theta + \frac{1}{2}\dot{\theta} + \frac{1}{\sqrt{2}}K) + \mathcal{E} \right] (\beta_\perp + \alpha)$$

$$- \frac{1}{\sqrt{2}} \left[\theta(\frac{1}{2}\theta - \frac{1}{2}\dot{\theta} + \frac{1}{\sqrt{2}}K) + \mathcal{D} + 8\pi T_{\mu\nu} k^\mu \acute{k}^\nu \right] (\beta_\perp - \alpha)$$

$$+ \theta s^i \bar{\nabla}_i \alpha$$

$$\mathcal{D} \equiv h^{ij} (D_i + J_i)(D_j + J_j) - \frac{1}{2} R$$

$$\mathcal{E} \equiv \sigma_{ij} \sigma^{ij} + 8\pi T_{\mu\nu} k^\mu k^\nu$$

$$\sigma_{ij} = \frac{1}{\sqrt{2}} (H_{ij} - \frac{1}{2} h_{ij} H) \left(1 - \frac{\beta_\perp}{\alpha} \right)$$

$$- \frac{1}{\sqrt{2}} \frac{\psi^4}{\alpha} \left\{ \tilde{D}_{(i} \beta_{\parallel j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \beta_{\parallel}^k - \frac{1}{2} [\tilde{h}_{ik} \tilde{h}_{jl} \tilde{u}^{kl} - \frac{1}{2} \tilde{h}_{ij} \tilde{h}_{kl} \tilde{u}^{kl}] \right\}$$

$$\tilde{s}^k \tilde{\nabla}_k \ln \psi = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J)$$

$$\beta^i = \alpha \psi^{-2} \tilde{s}^i + \beta_{\parallel}^i$$

$$0 = \tilde{D}_{(i} \beta_{\parallel j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \beta_{\parallel}^k$$

$$\partial_t \ln \psi = \left[\tilde{D}_k \beta_{\parallel}^k + 4\beta_{\parallel}^k \tilde{D}_k \ln \psi \right.$$

$$\left. - \frac{1}{2} \tilde{h}_{kl} \tilde{u}^{kl} \right.$$

$$\left. - +\sqrt{2}\theta + (\beta_\perp - \alpha)H \right]$$

Defining the Spin of the Black Hole

The spin parameters β_{\parallel}^i can be defined by demanding that the time vector associated with quasi-equilibrium in the corotating frame must be null, forming the null generators of the horizon.

$$k^{\mu} \propto (n^{\mu} + s^{\mu}) \implies k^{\mu} = [1, \alpha s^i - \beta^i]$$

This vector k^{μ} is null for any choice of α & β^i .

In the frame where a black hole is not spinning, the null time vector has components $t^{\mu} = [1, \vec{0}]$.

Corotating Holes

Corotating holes are at rest in the corotating frame, where we must pose boundary conditions. So,

$$k^{\mu} = [1, \alpha s^i - \beta^i] = [1, \vec{0}]$$

Thus we find

$$\beta^i = \alpha s^i \implies \beta_{\parallel}^i = 0$$

Irrotational Holes

Irrotational holes are at rest in the inertial frame. With the time vectors in the inertial and corotating frames related by

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} + \Omega \frac{\partial}{\partial \phi}$$

$$k^{\mu} = [1, \alpha s^i - \beta^i] = [1, -\Omega(\partial/\partial\phi)^i]$$

Thus we find

$$\beta^i = \alpha s^i + \Omega \left(\frac{\partial}{\partial \phi} \right)^i \implies \beta_{\parallel}^i = \Omega \xi^i$$

$$\xi^i \approx \left(\frac{\partial}{\partial \phi} \right)^i \quad \& \quad \tilde{D}_{(i} \xi_{j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \xi^k = 0$$

The Lapse BC & QE

So far, nothing has fixed a boundary condition on the lapse α . One possibility[6] is to recall that $\theta\acute{\theta}$ is a Lorentz invariant and so to consider $\mathcal{L}_\zeta\theta\acute{\theta} = 0$ as a quasi-equilibrium condition.

$$\mathcal{L}_\zeta\theta\acute{\theta} = 0 \quad \Rightarrow \quad J\tilde{s}^i\tilde{\nabla}_i\alpha = -\psi^2(J^2 - JK + \tilde{D})\alpha$$

$$\tilde{D} \equiv \psi^{-4}[\tilde{h}^{ij}(\tilde{D}_i - J_i)(\tilde{D}_j - J_j) - \frac{1}{2}\tilde{R} + 2\tilde{D}^2 \ln \psi]$$

This conditions is satisfied for stationary solutions, but seems to be degenerate with the other QE boundary conditions. To see this, note that the stationary maximal slicings of Schwarzschild form a 1-parameter family:

$$ds^2 = \frac{dR^2}{1 - \frac{2M}{R} + \frac{C^2}{R^4}} + R^2 d^2\Omega$$

$$\alpha = \sqrt{1 - \frac{2M}{R} + \frac{C^2}{R^4}}$$

$$\beta^R = \frac{C}{R^2} \sqrt{1 - \frac{2M}{R} + \frac{C^2}{R^4}}$$

$$K_j^i = \frac{C}{R^3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\alpha|_S = \frac{C}{4M^2}$$

The Orbital Angular Velocity

- For a given choice of the Lapse BC, $\tilde{\gamma}_{ij}$ and K , we are still left with a family of solutions parameterized by the orbital angular velocity Ω .
- Except for the case of a single spinning black hole, it is not reasonable to expect more than one value of Ω to correspond to a system in quasi-equilibrium.

GGB[7, 8] have suggested a way to pick the quasi-equilibrium value of Ω :

Ω is chosen as the value for which the ADM energy E_{ADM} equals the Komar mass M_{K} .

Komar mass	$M_{\text{K}} = \frac{1}{4\pi} \oint_{\infty} \gamma^{ij} (\bar{\nabla}_i \alpha - \beta^k K_{ik}) d^2 S_j$	Acceptable definition of the mass <i>only for stationary spacetimes.</i>
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ADM energy	$E_{\text{ADM}} = \frac{1}{16\pi} \oint_{\infty} \gamma^{ij} \bar{\nabla}_k (\mathcal{G}_i^k - \delta_i^k \mathcal{G}) d^2 S_j$	Acceptable definition of the mass <i>for arbitrary spacetimes.</i> $\mathcal{G}_{ij} \equiv \gamma_{ij} - f_{ij}$
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Summary of QE Formalism

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \gamma^{ij} K \quad \tilde{A}^{ij} = \frac{\psi^6}{2\alpha} (\tilde{\mathbb{L}}\beta)^{ij} \quad \partial_t \tilde{\gamma}_{ij} = 0$$

$$\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0$$

$$\tilde{\nabla}_j (\tilde{\mathbb{L}}\beta)^{ij} - (\tilde{\mathbb{L}}\beta)^{ij} \tilde{\nabla}_j \ln \alpha \psi^{-6} = \frac{4}{3} \alpha \tilde{\nabla}^i K$$

$$\tilde{\nabla}^2 (\alpha \psi) - (\alpha \psi) \left[\frac{1}{8} \tilde{R} + \frac{5}{12} \psi^4 K^2 + \frac{7}{8} \psi^{-8} A_{ij} A^{ij} \right] = \psi^5 \beta^i \tilde{\nabla}_i K \quad \partial_t K = 0$$

$$\tilde{s}^k \tilde{\nabla}_k \ln \psi|_S = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J)|_S \quad \theta = 0$$

$$\beta^i|_S = \begin{cases} \alpha \psi^{-2} \tilde{s}^i|_S & \text{corotation} \\ \alpha \psi^{-2} \tilde{s}^i|_S + \Omega \xi^i|_S & \text{irrotation} \end{cases} \quad \begin{matrix} \mathcal{L}_\zeta \theta = 0 \\ \sigma_{ij} = 0 \end{matrix}$$

$\alpha|_S = \text{unspecified by QE}$

$$\psi|_{r \rightarrow \infty} = 1$$

$$\beta^i|_{r \rightarrow \infty} = \Omega \left(\frac{\partial}{\partial \phi} \right)^i$$

$$\alpha|_{r \rightarrow \infty} = 1$$

The only remaining freedom in the system is the choice of the lapse boundary condition, the initial spatial and temporal gauge, and the initial dynamical (“wave”) content found in

$\alpha|_S$, $\tilde{\gamma}_{ij}$ and K .

Results

Corotation

$\tilde{\gamma}_{ij} = f_{ij}$: Maximal Slicing:

- $\frac{\partial(\alpha\psi)}{\partial r} = 0$
- $\alpha\psi = \frac{1}{2}$
- $\frac{\partial(\alpha\psi)}{\partial r} = \frac{\alpha\psi}{2r}$

$\tilde{\gamma}_{ij} = f_{ij}$: Eddington-Finkelstein Slicing:

- $\frac{\partial(\alpha\psi)}{\partial r} = 0$
- $\alpha\psi = \frac{1}{2}$
- $\frac{\partial(\alpha\psi)}{\partial r} = \frac{\alpha\psi}{2r}$

Compared with

- Effective-One-Body PN[5]
- Inversion-Symmetric HKV[8]

Irrotation

$\tilde{\gamma}_{ij} = f_{ij}$: Maximal Slicing:

- $\frac{\partial(\alpha\psi)}{\partial r} = 0$
- $\alpha\psi = \frac{1}{2}$
- $\frac{\partial(\alpha\psi)}{\partial r} = \frac{\alpha\psi}{2r}$

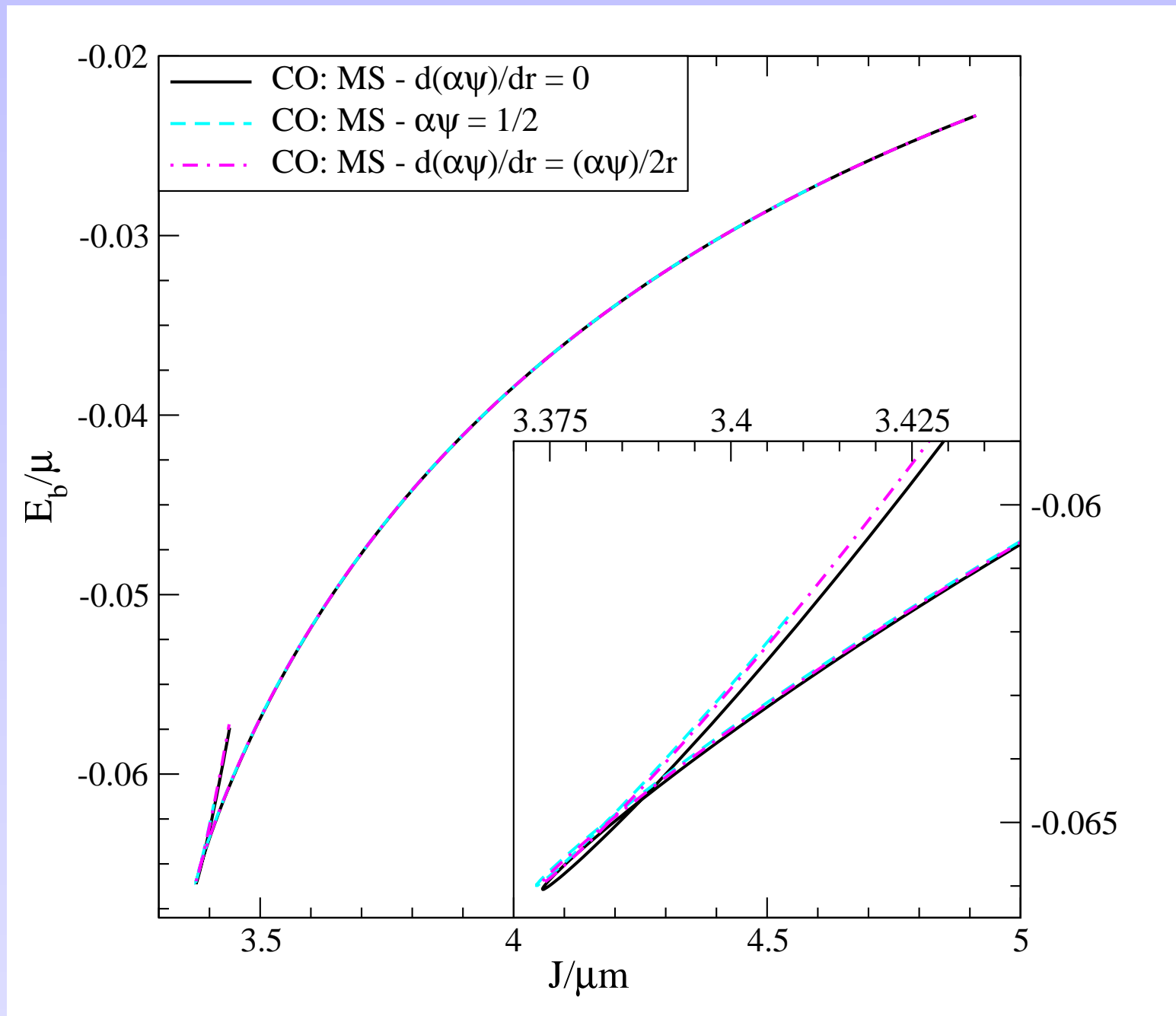
$\tilde{\gamma}_{ij} = f_{ij}$: Eddington-Finkelstein Slicing:

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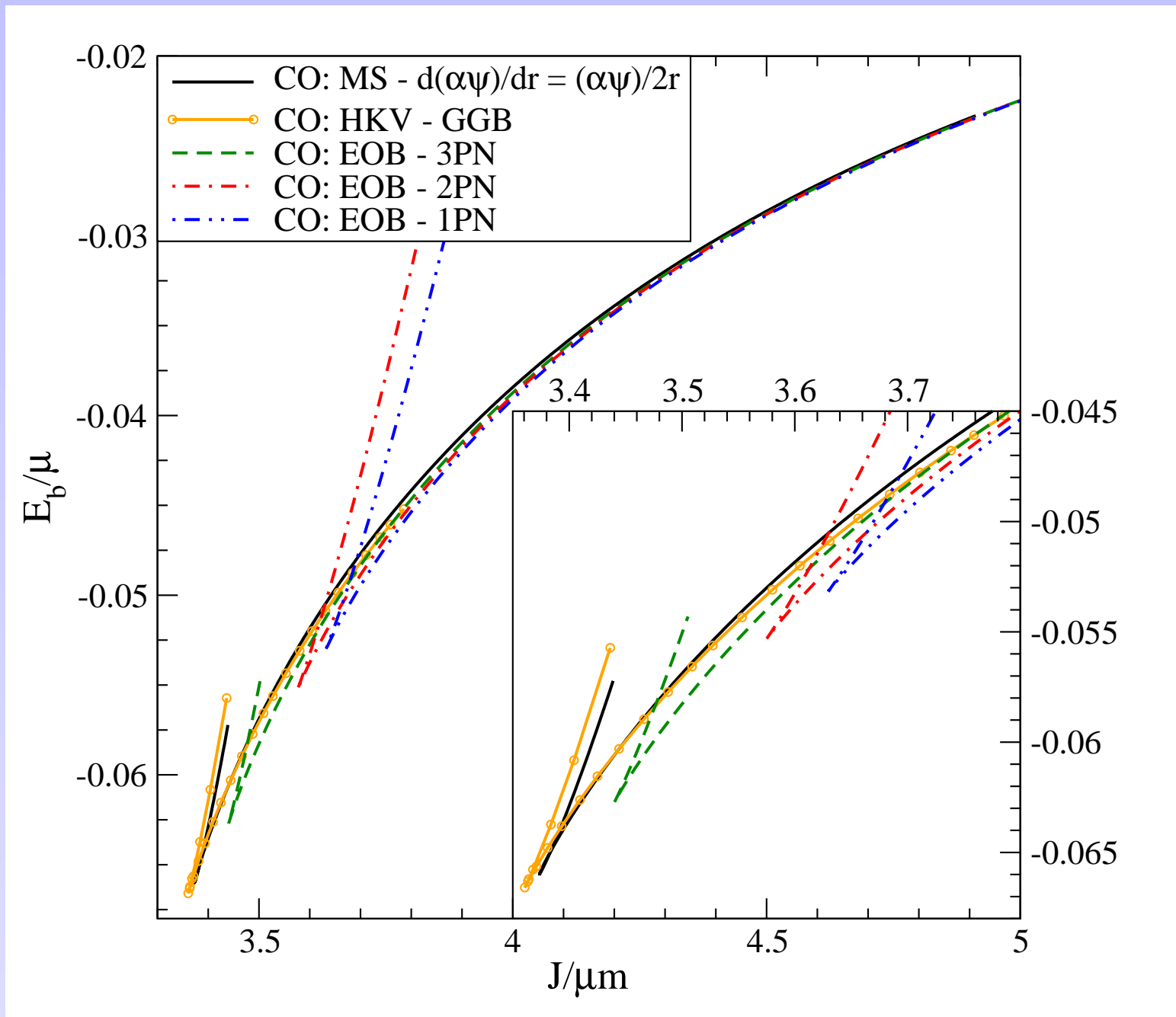
Compared with

- Effective-One-Body PN[5]
- Conformal Imaging[4]
- Puncture Method[1]

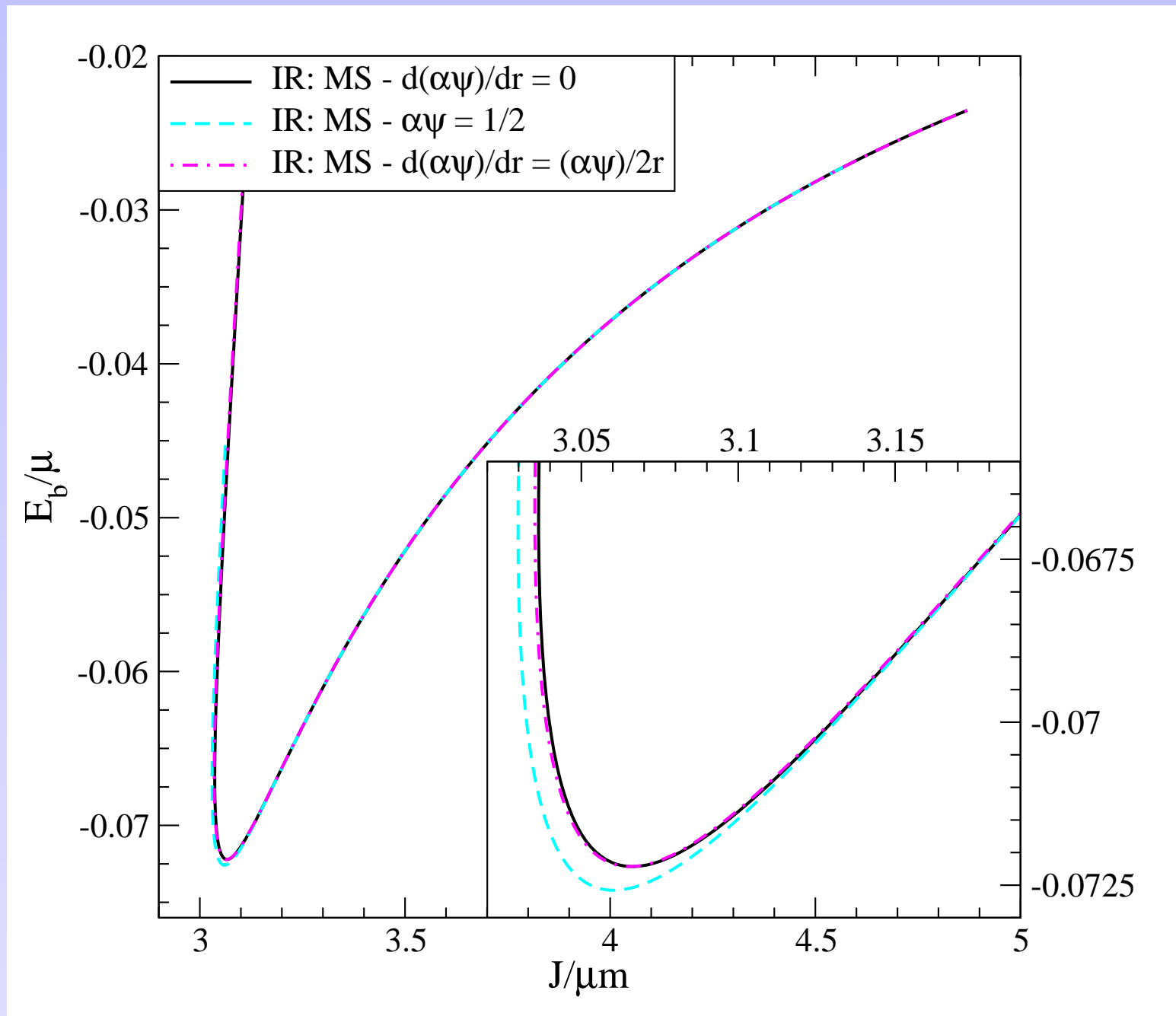
Corotating; Maximal Slicce; QE-BC; E_b/μ vs $J/\mu m$



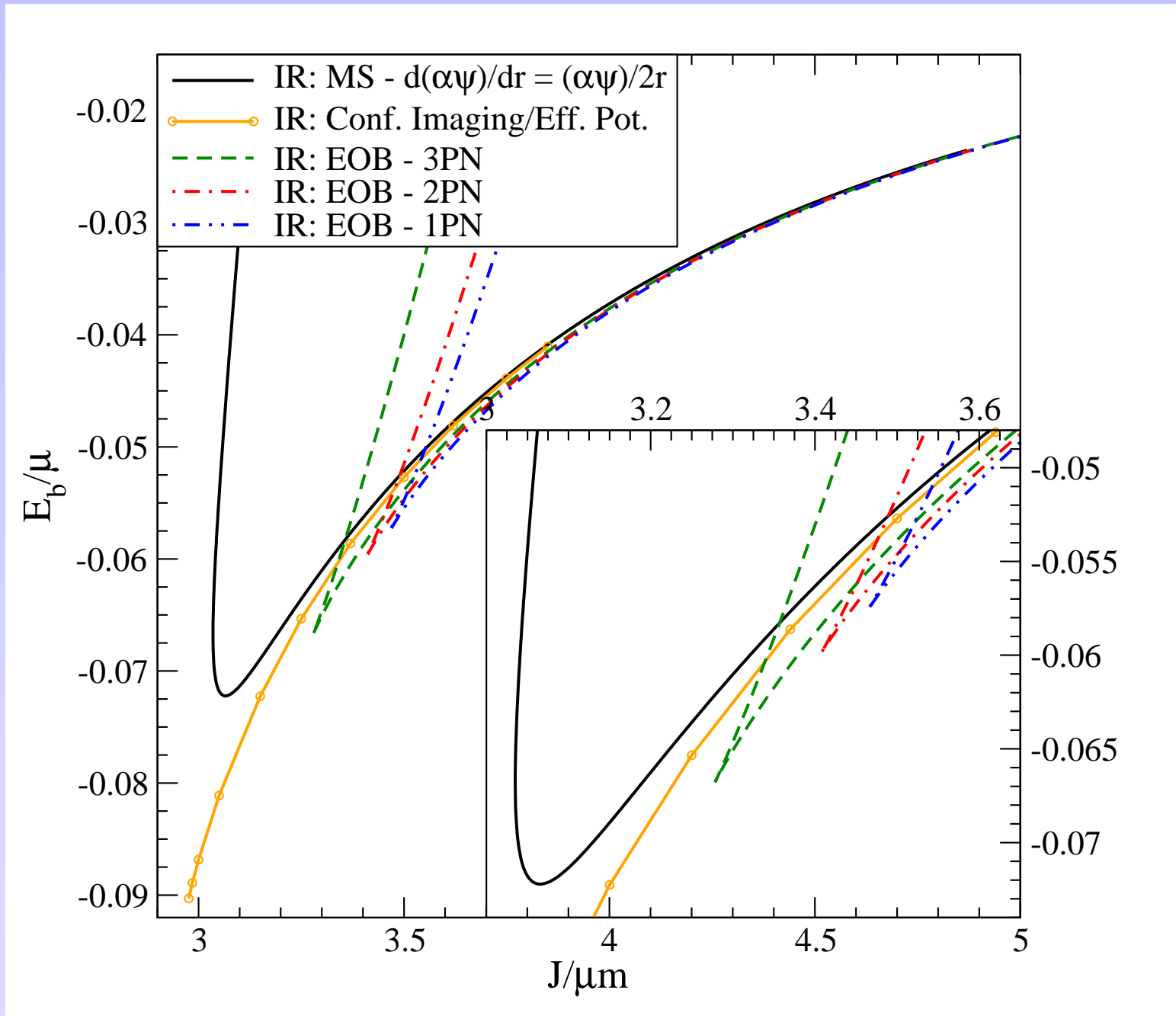
Corotating; Maximal Slicce; Comparison; E_b/μ vs $J/\mu m$



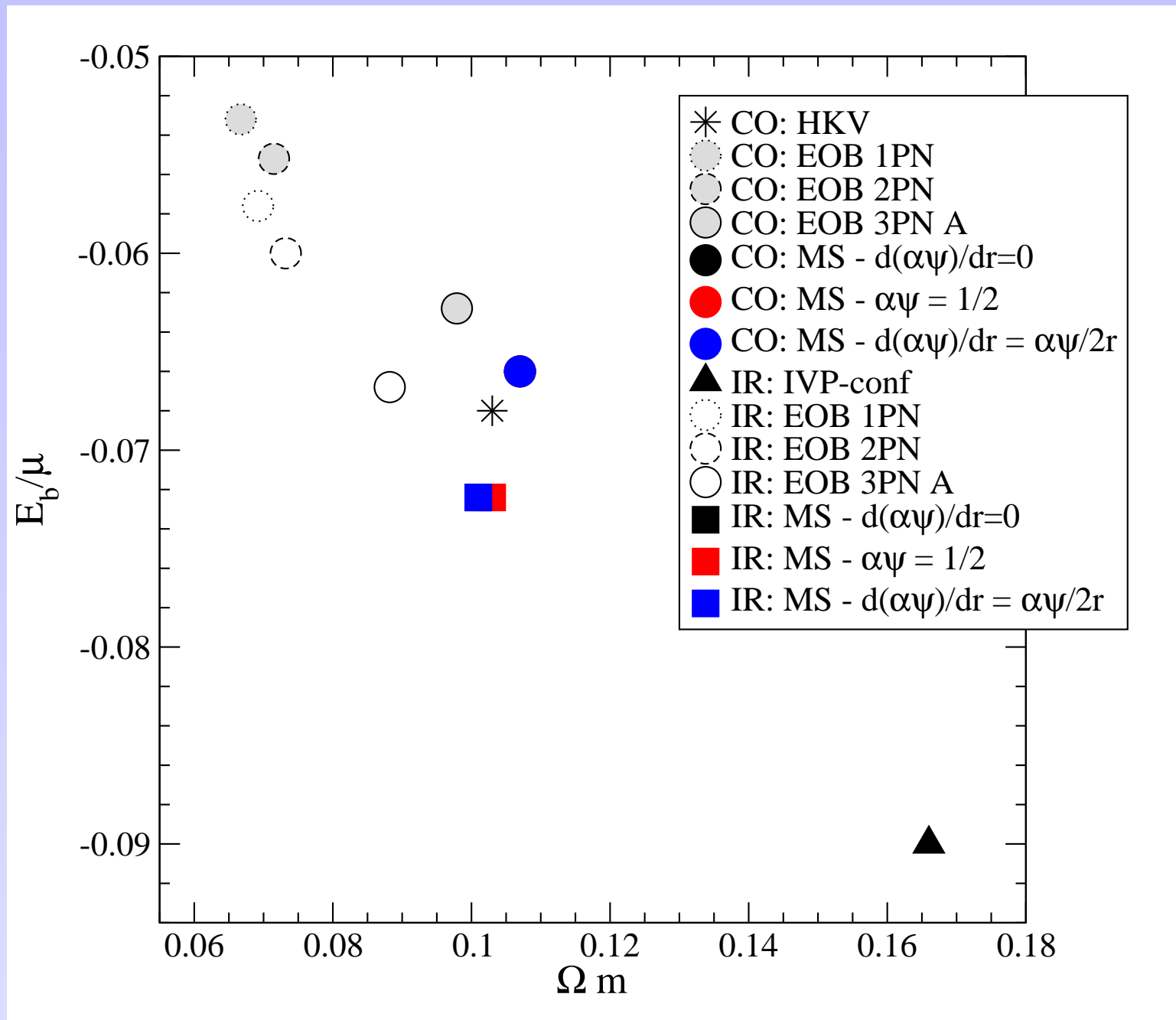
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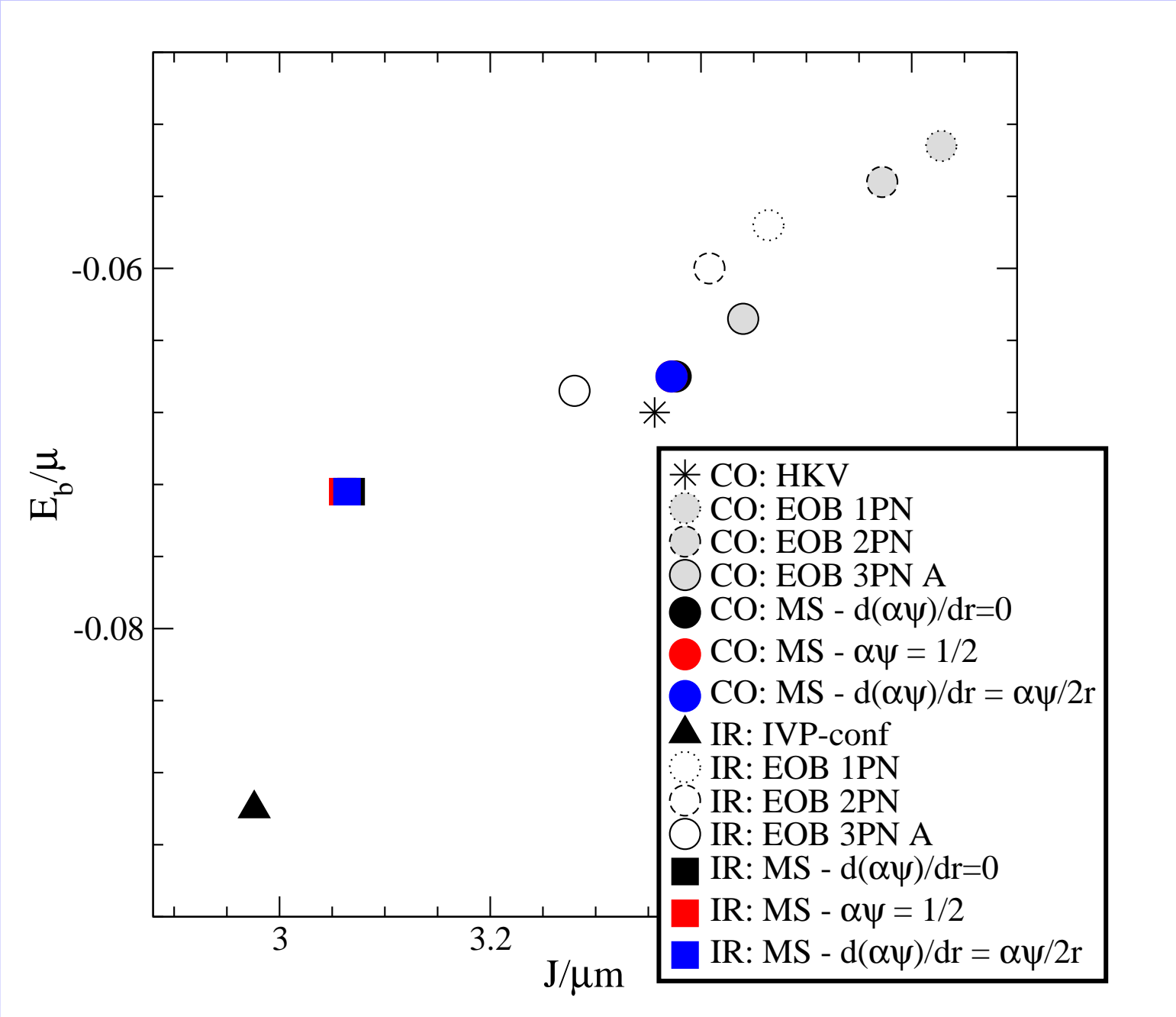
Irrotational; Maximal Slicce; Comparison; E_b/μ vs $J/\mu m$



Maximal Slicce; Comparison of ISCO; E_b/M_{irr} vs ΩM_{irr}



Maximal Slicce; Comparison of ISCO; E_b/M_{irr} vs J/M_{irr}^2



Open Questions

- Is the physics of the corotating and irrotational models correct?
 - Do the corotating black holes have the correct angular momentum?
 - Is the angular momentum of the irrotational holes nearly zero?
- How do we make a *physically motivated* choice for $\tilde{\gamma}_{ij}$?

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