Approximate Killing Vectors and Black-Hole Diagnostics

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Measuring the Spin of a Black Hole

- Spin is only rigorously defined at spatial/null infinity.
- Must use *quasi-local* definition: e.g. Brown & York[2] or Ashtekar & Krishnan[1]

$$S = -\frac{1}{8\pi}\oint_{BH} K_{ij}\xi^i s^j \sqrt{h} d^2x$$

$$\xi^i = \begin{cases} \xi^i_{\mathsf{CK}} & : \text{ Killing vector of } \tilde{h}_{ij} \Rightarrow \text{conformal Killing vector of } h_{ij} \\ \xi^i_{\mathsf{AKV}} & : \text{ Approximate Killing vector of } h_{ij} \end{cases}$$

Approximate Killing Vectors

Killing Transport on S^2[3]

$$D_i \xi_j = \epsilon_{ij} L$$

$$D_i L = -\frac{1}{2}{}^2R \epsilon_i{}^j \xi_j$$

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- Equations assume KV exists!
- Solution is "path dependent"
- Solution violates equations
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New AKV Method

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\begin{aligned}
\xi^i &= \epsilon^{ij}D_jv + D^id \\
D_i\xi_j &= \epsilon_{ij}L + S_{ij} + h_{ij}\Lambda
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- Find solutions that minimize S^{ij}!
- $\xi^i = \epsilon^{ij}D_jv$ by construction
- $L = \frac{1}{2}\epsilon_{ij}D^i\xi^j$ by construction

New Approximate Killing Vectors

- Minimize

$$S_{ij}S^{ij} = (D_i D_j v)(D^i D^j v) - \tfrac{1}{2}(D^k D_k v)^2$$

subject to constraint that $|\vec{\xi}|^2 = (D_i v)(D^i v) =$ const.

- $$\mathcal{L} \equiv S_{ij}S^{ij} + \tfrac{1}{2}{}^2\!R\Theta(D_k v)(D^k v)$$

New Approximate Killing Vectors

- Minimize

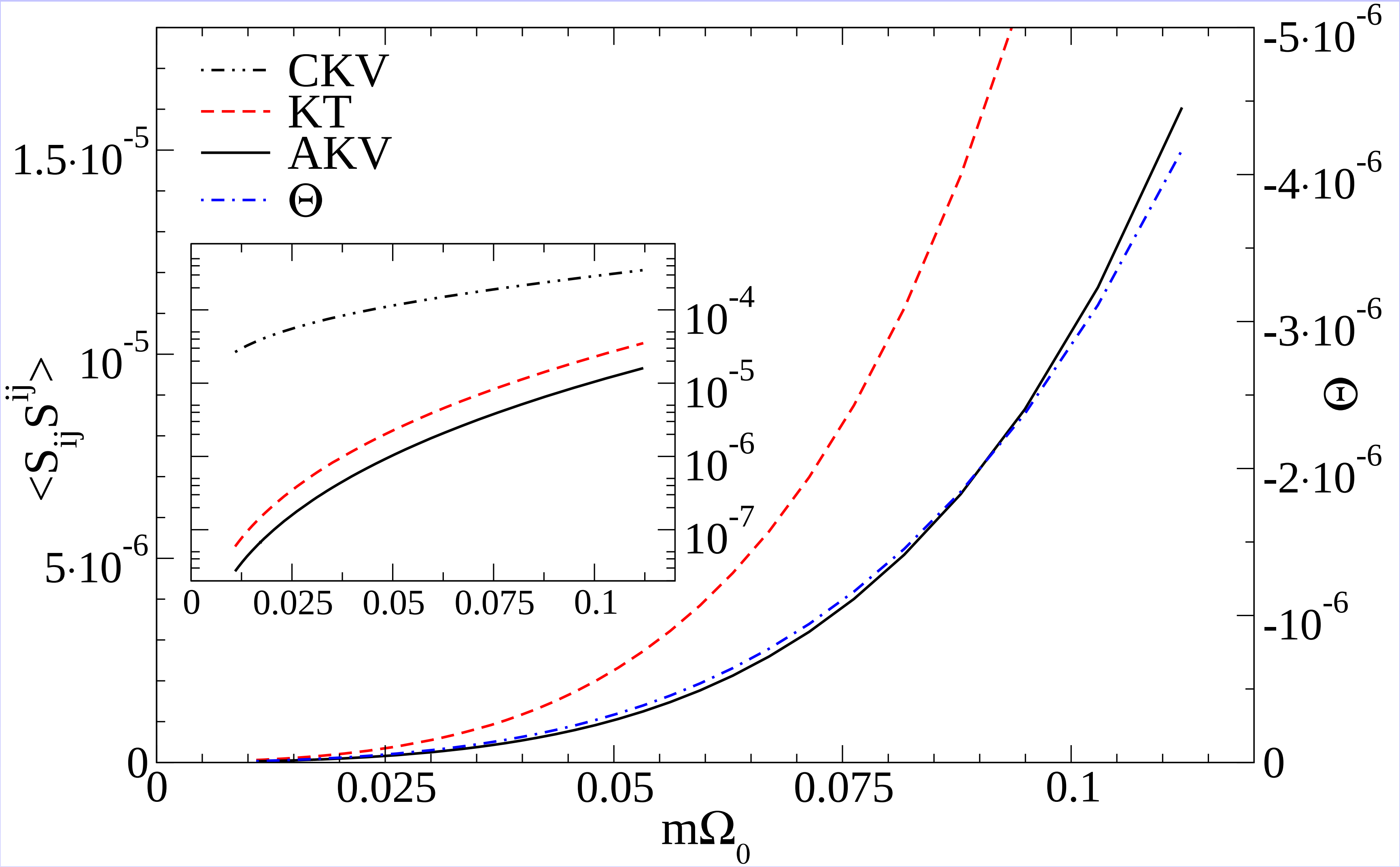
$$S_{ij}S^{ij} = (D_iD_jv)(D^iD^jv) - \tfrac{1}{2}(D^kD_kv)^2$$

subject to constraint that $|\vec{\xi}|^2 = (D_iv)(D^iv) = \text{const.}$

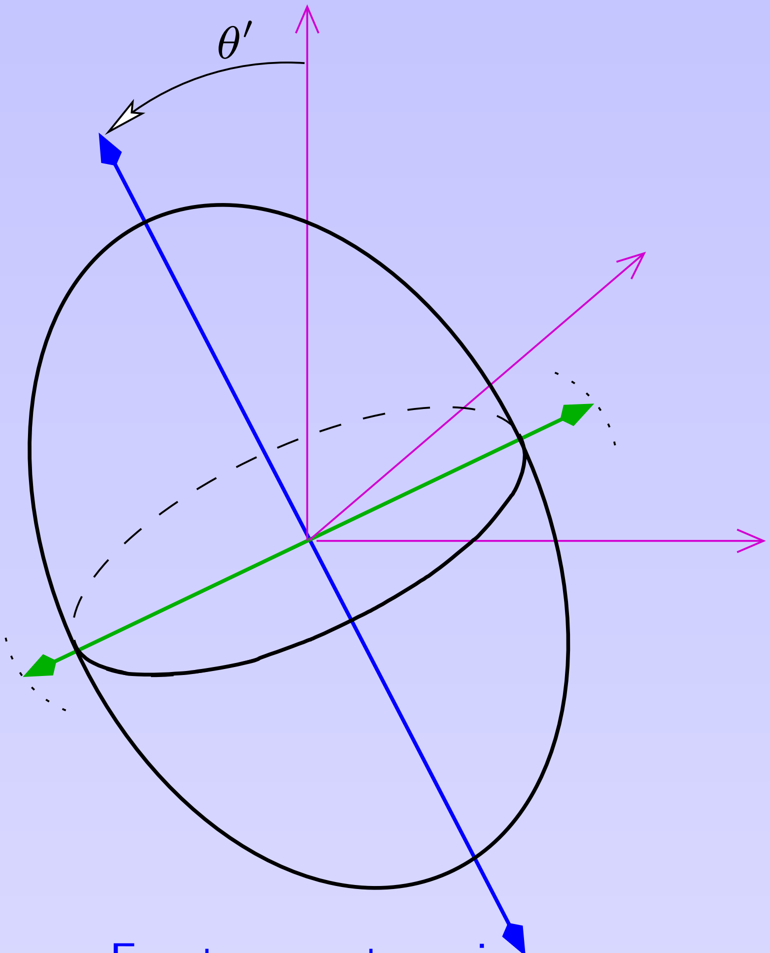
- $$\mathcal{L} \equiv S_{ij}S^{ij} + \tfrac{1}{2}{}^2R\Theta(D_kv)(D^kv)$$

$$\frac{\delta\mathcal{L}}{\delta v} = 0 \quad \Rightarrow \quad \begin{aligned} \xi^i &\equiv \epsilon^{ij}D_jv \\ D^iD_iv + 2L &= 0 \\ D^iD_iL - (1-\Theta)\left[\tfrac{1}{2}(D^{i\,2}R)D_iv - {}^2RL\right] &= 0 \end{aligned}$$

Corotating BBH “Spin AKVs”



Tests With Exact Symmetry



Exact symmetry axis

Tests

$$\mathrm{d}s^2 = \psi^4(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

$$\psi(\theta,\phi) = A + B\sum_{m=-2}^{2} Y_{2m}(\theta,\phi)Y^*_{2m}(\theta',\phi')$$

- Finds Killing vectors when they exist ($\Theta = 0$).
- Finds continuum of additional approximate solutions ($\Theta \neq 0$) when exact symmetry is present.

Corotating BBH AKVs

In addition to expected "spin AKV," we find 2 more AKVs.

No spin!

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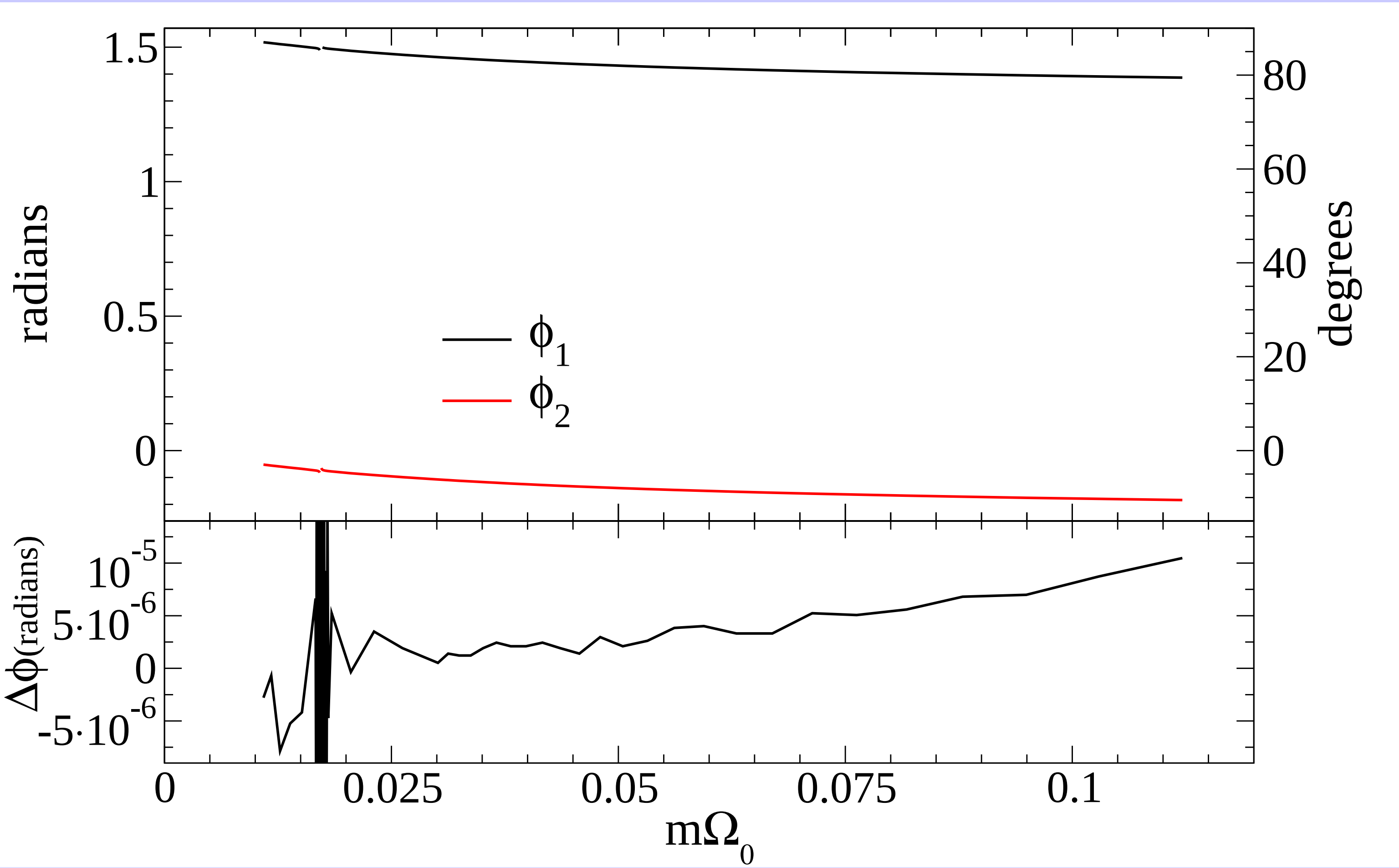
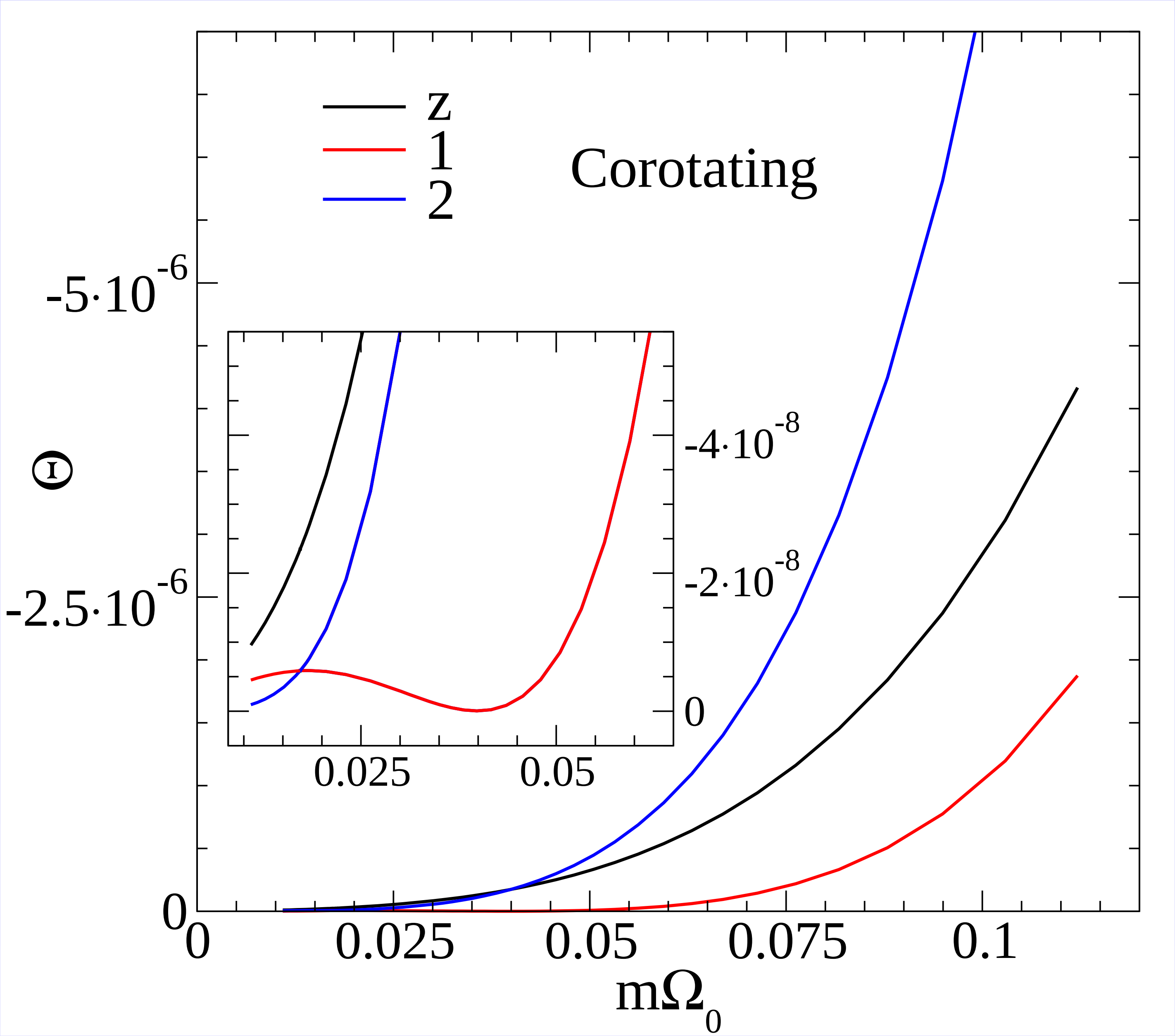


Corotating BBH AKVs



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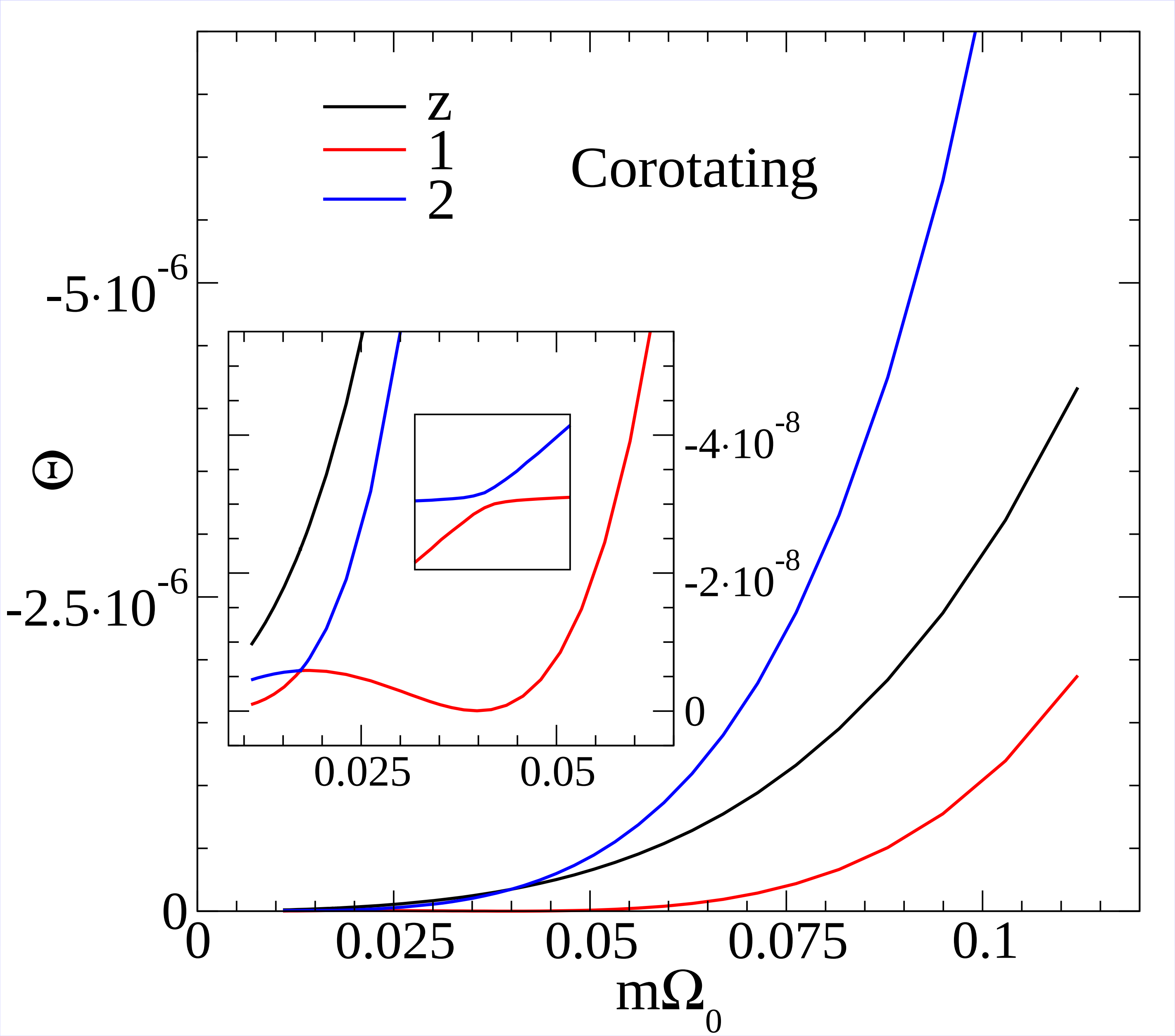

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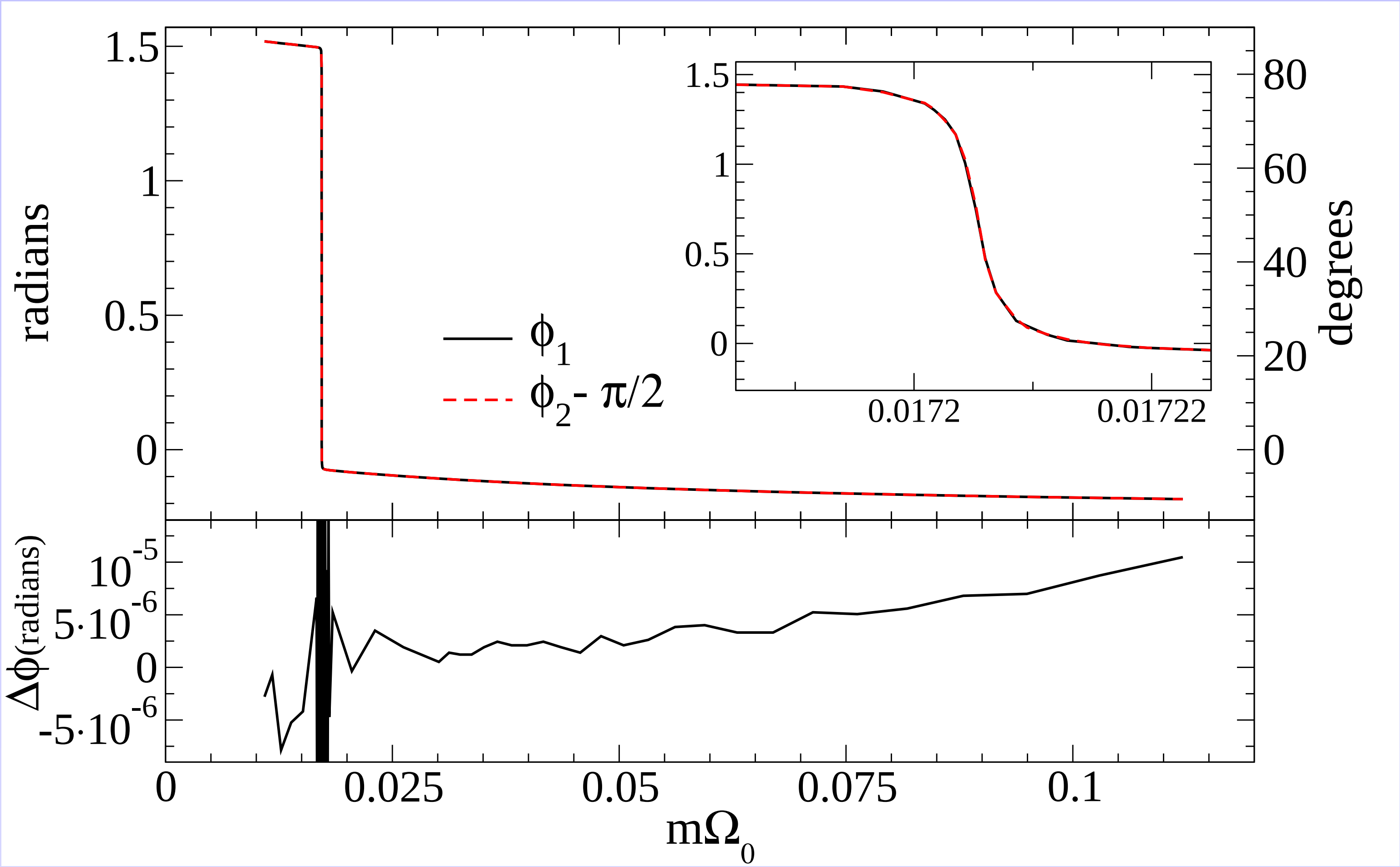

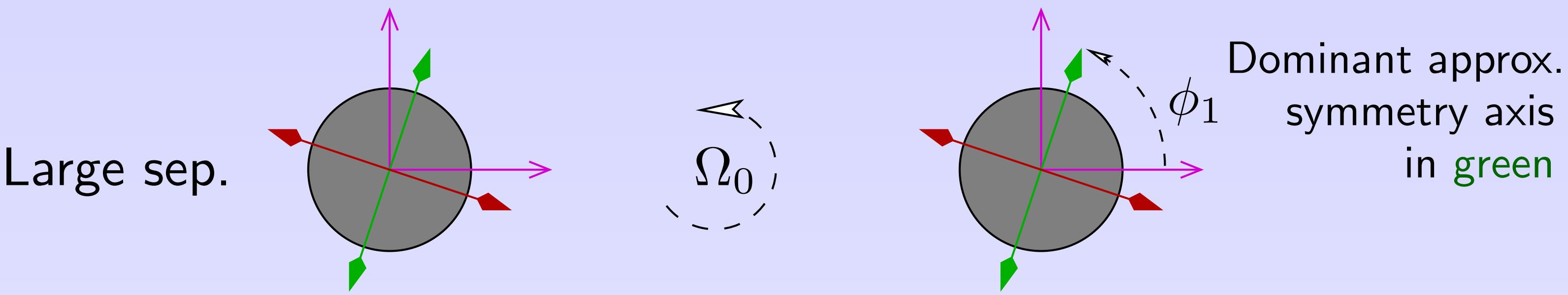

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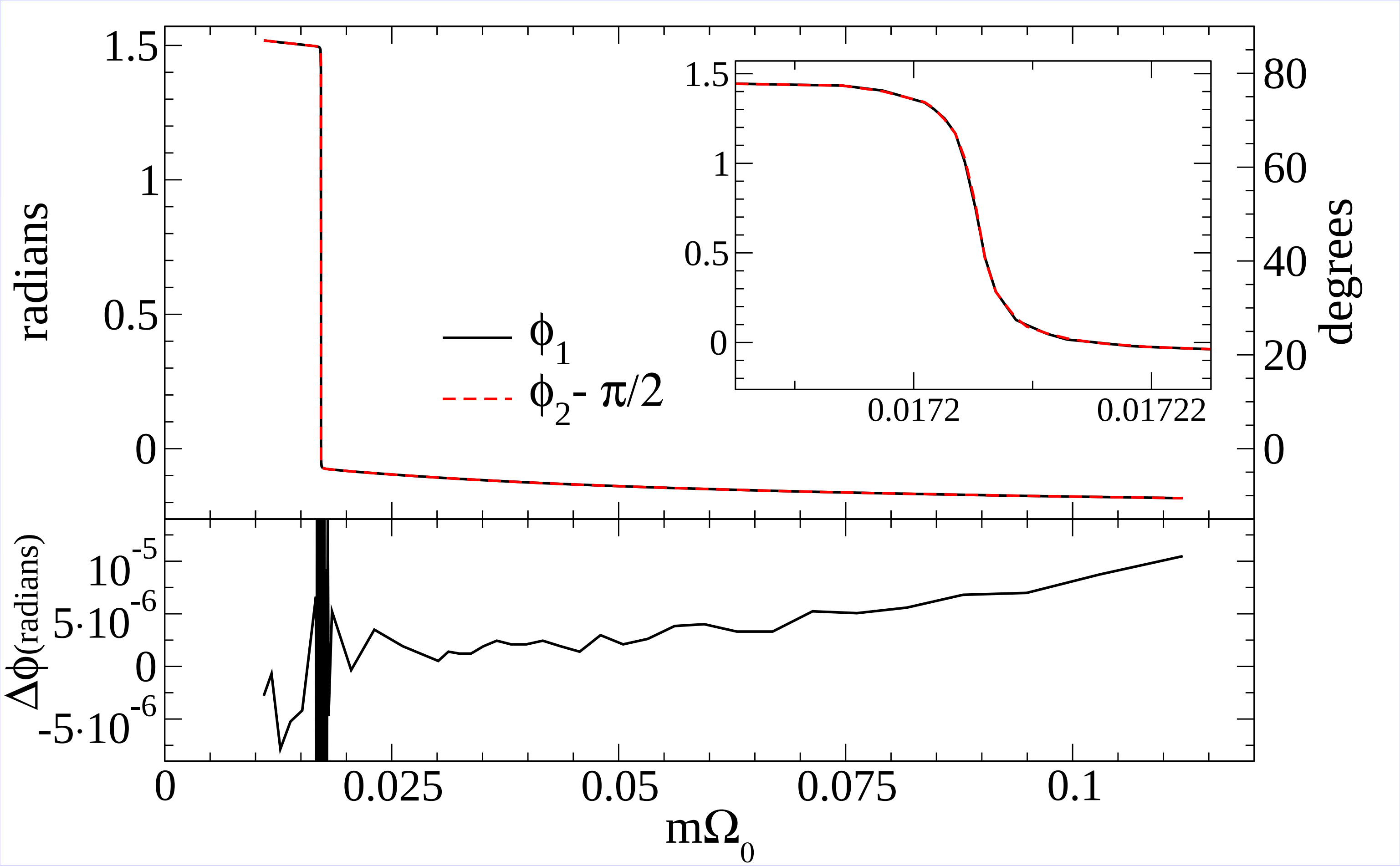
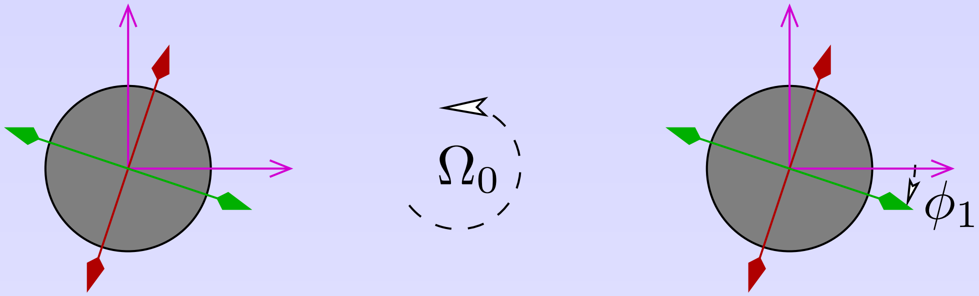
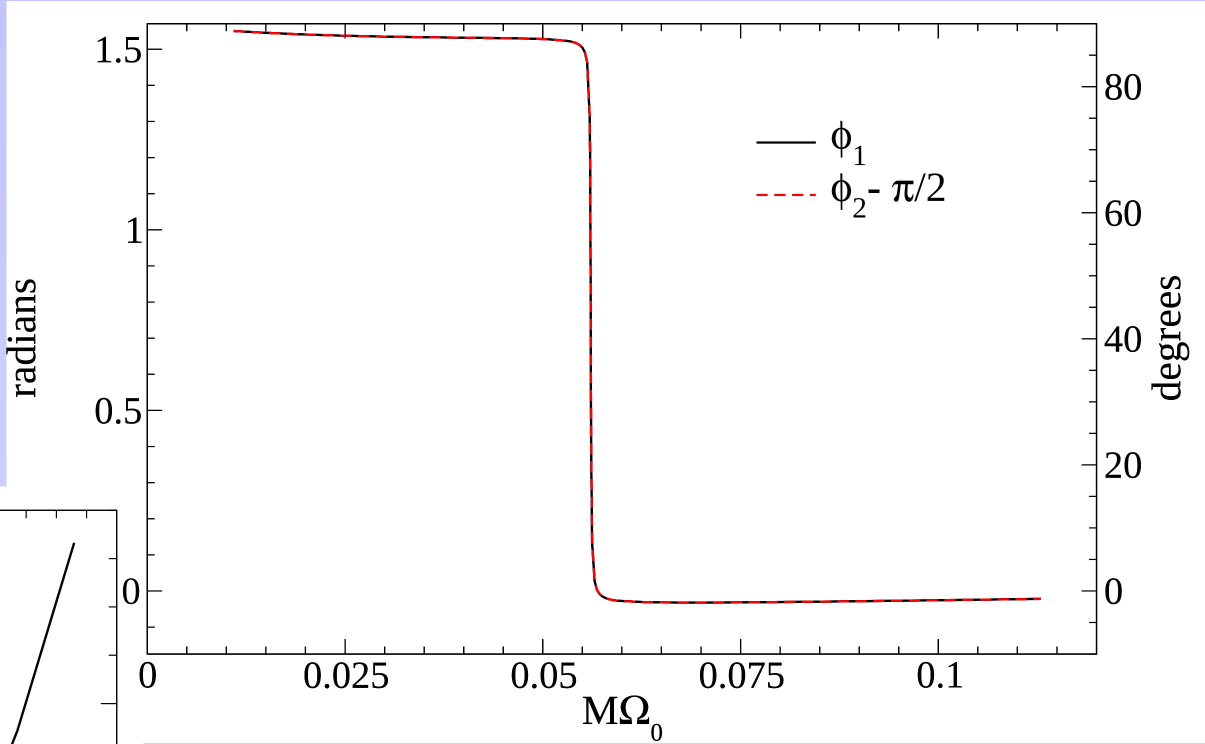


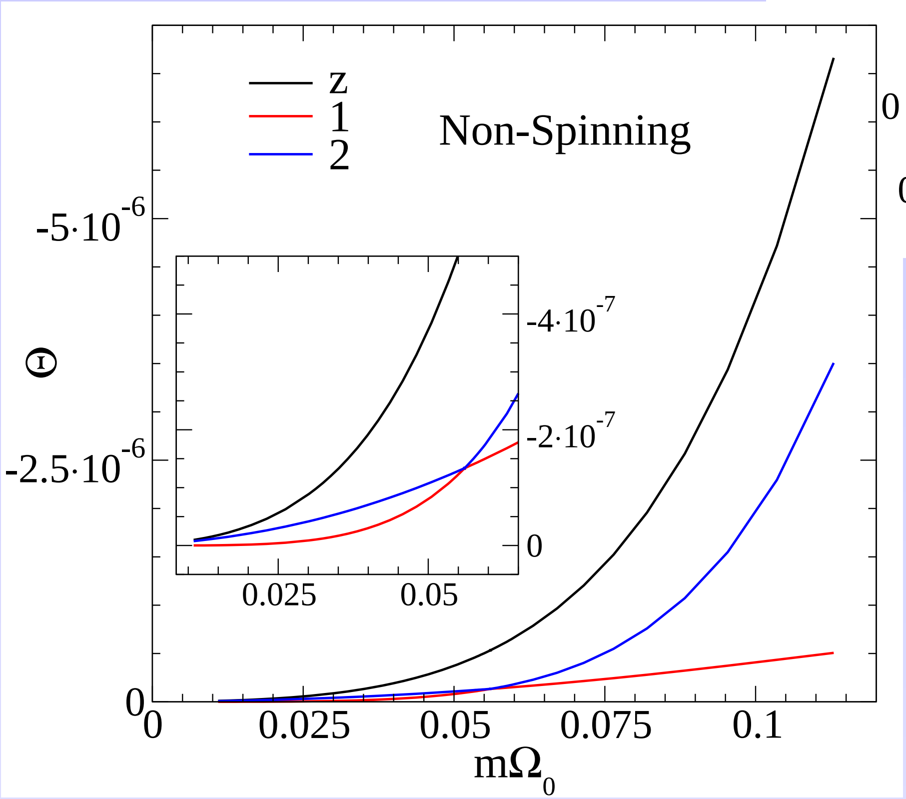
Small sep.



Dominant approx. symmetry axis in green

Non-Spinning BBH AKVs





Summary

- New method determines *best* AKV:

  smallest $S_{ij}S^{ij} : \xi^i = \epsilon^{ij} D_j v$.

- Computed spin essentially the same as from Killing Transport for corotation & non-spinning equal-mass cases. Differences may be more significant when higher spin rates or greater BH distortion are considered.

- Only one AKV solution yields spin. Other AKV solutions provide new diagnostic tool for exploring horizon geometry of distorted BHs.

- Algorithm for "conformal spheres" implemented and tested. Algorithm for general S^2 nearly finished.

References

[1] A. Ashtekar and B. Krishnan. Dynamical horizons and their properties. *Phys. Rev. D*, 68:104030/1–25, 2003. 1

[2] J. D. Brown and J. W. York, Jr. Quasilocal energy and conserved charges derived from the gravitational action. *Phys. Rev. D*, 47:1407–1419, 1993. 1

[3] O. Dreyer, B. Krishnan, D. Shoemaker, and E. Schnetter. Introduction to isolated horizons in numerical relativity. *Phys. Rev. D*, 67:024018/1–14, Jan. 2003. 2