

Generating Binary Black Hole Initial Data

Gregory B. Cook

Wake Forest University

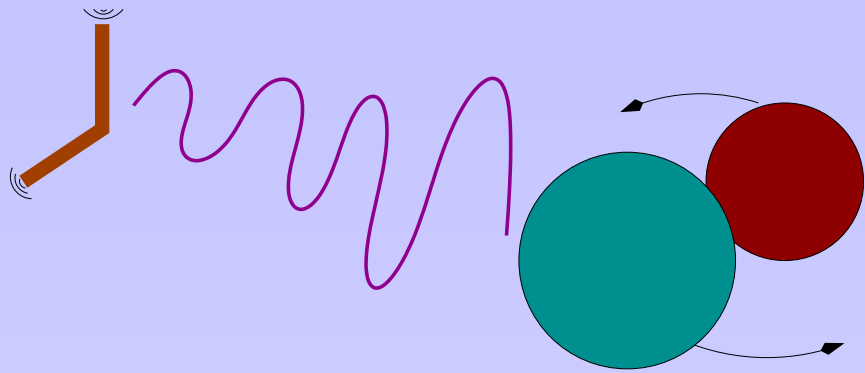
May 2, 2003

Abstract

A formalism for constructing initial data representing black-hole binaries in quasi-equilibrium is developed. If each black hole is assumed to be in quasi-equilibrium, then conditions on the shear and expansion of the outgoing null rays at each black hole yield a set of boundary conditions for the initial data variables. This formalism should allow for the construction of completely general quasi-equilibrium black hole binary initial data. Initial numerical results using this approach will be examined and compared with previous numerical and post-Newtonian results.

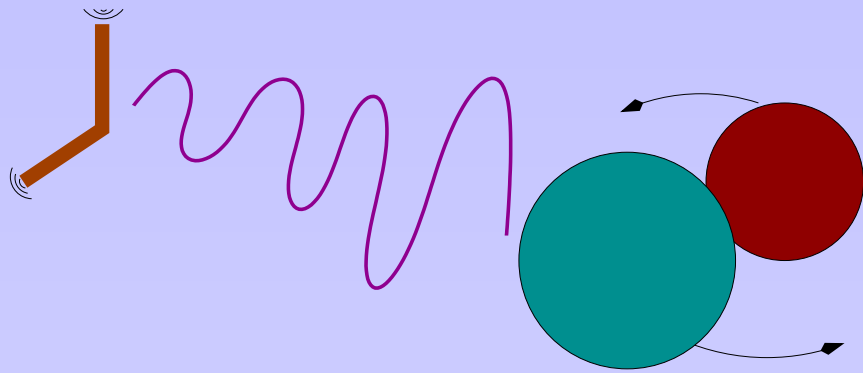
Collaborators: Harald Pfeiffer & Saul Teukolsky (Cornell)

Motivation



- Black hole binaries are among the most likely sources for early detection with LIGO, VIRGO, GEO, . . .
- Available computed waveforms should increase chance of detecting collision events.

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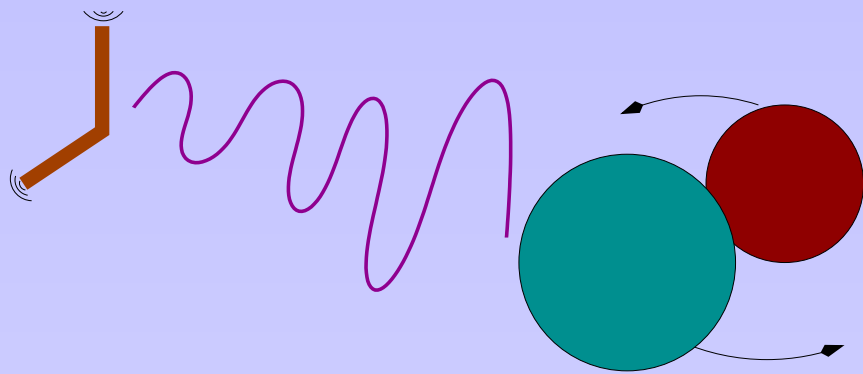


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Quasi-Equilibrium Binary Data

- General Relativity doesn't permit *true* equilibrium for astrophysical binary systems.
- When the bodies are sufficiently far apart, the timescale for orbital decay is much larger than the orbital period.
- If the orbit is nearly circular (*quasi-circular*) then there is a *corotating reference frame* in which the binary appears to be at rest.

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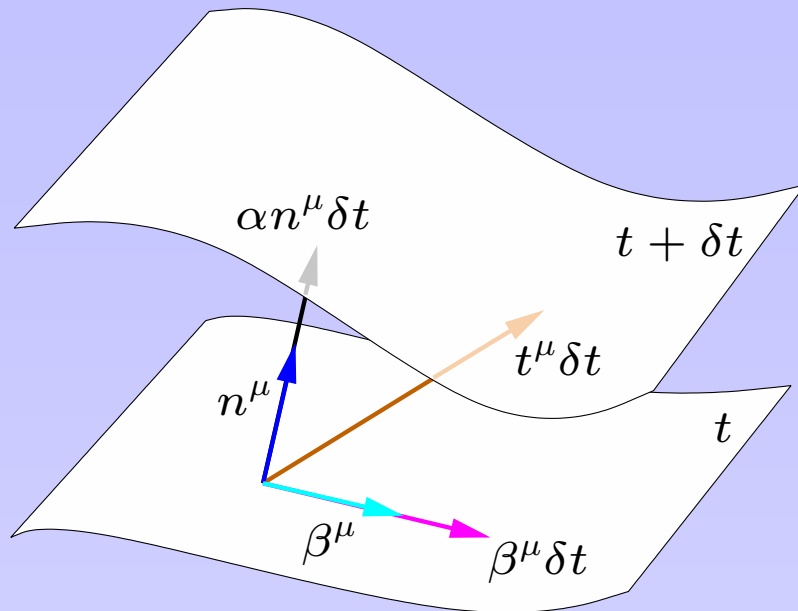


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- If the orbit is nearly circular (*quasi-circular*) then there is a *corotating reference frame* in which the binary appears to be at rest.
- ★ Quasi-equilibrium gives us a *physical condition* to guide us in fixing boundary conditions and data that is not otherwise constrained.

The 3 + 1 Decomposition



Lapse : α

Spatial metric : γ_{ij}

Shift vector : β^i

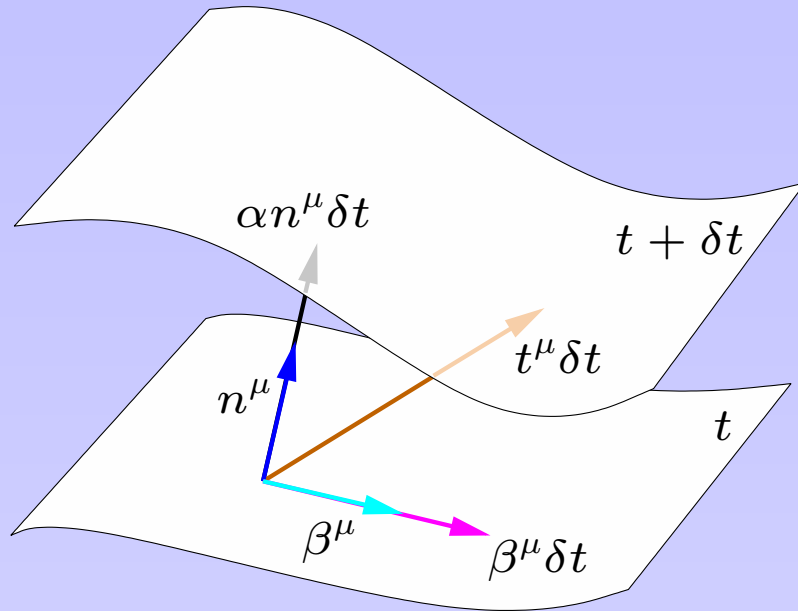
Extrinsic Curvature : K_{ij}

Time vector : $t^\mu = \alpha n^\mu + \beta^\mu$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \quad K_{\mu\nu} = -\frac{1}{2}\gamma_\mu^\alpha \gamma_\nu^\beta \mathcal{L}_n g_{\alpha\beta}$$

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Constraint equations

$$\bar{R} + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

$$\bar{\nabla}_j (K^{ij} - \gamma^{ij}K) = 8\pi j^i$$

$$S_{\mu\nu} \equiv \gamma_\mu^\alpha \gamma_\nu^\beta T_{\alpha\beta}$$

$$j_\mu \equiv -\gamma_\mu^\nu n^\alpha T_{\nu\alpha}$$

$$\rho \equiv n^\mu n^\nu T_{\mu\nu}$$

$$T_{\mu\nu} = S_{\mu\nu} + 2n_{(\mu}j_{\nu)} + n_\mu n_\nu \rho$$

Evolution equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i$$

$$\partial_t K_{ij} = -\bar{\nabla}_i \bar{\nabla}_j \alpha + \alpha \left[\bar{R}_{ij} - 2K_{il}K_j^\ell + K K_{ij} - 8\pi S_{ij} + 4\pi \gamma_{ij}(S - \rho) \right]$$

$$+ \beta^\ell \bar{\nabla}_\ell K_{ij} + K_{il} \bar{\nabla}_j \beta^\ell + K_{jl} \bar{\nabla}_i \beta^\ell$$

Generalized Conformal/TT Initial-Data Decomposition[13]

$$\begin{aligned}
 \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad : \quad K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \psi^{-4} \tilde{\gamma}^{ij} K \quad : \quad \tilde{A}^{ij} &\equiv \frac{1}{\tilde{\sigma}} (\tilde{\mathbb{L}}V)^{ij} + \tilde{M}^{ij} \\
 \tilde{\sigma} &\equiv \psi^{-6} \sigma \quad : \quad \tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2\pi \psi^5 \rho \\
 \tilde{\nabla}_j (\tilde{\mathbb{L}}V)^{ij} - (\tilde{\mathbb{L}}V)^{ij} \tilde{\nabla}_j \ln \tilde{\sigma} - \frac{2}{3} \tilde{\sigma} \psi^6 \tilde{\nabla}^i K &= -\tilde{\sigma} \tilde{\nabla}_j \tilde{M}^{ij} + 8\pi \tilde{\sigma} \psi^{10} j^i
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\tilde{M}^{ij} is symmetric-tracefree, but not divergenceless. The variable V^i incorporates the solution of the constraints and the decomposition of \tilde{M}^{ij} into \tilde{M}_{TT}^{ij} .

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Conformal Thin Sandwich(TS)[16]

$$[\tilde{u}^{ij} \equiv -\partial_t \tilde{\gamma}^{ij}]$$

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α and β^i are the lapse and shift. \tilde{u}^{ij} is symmetric-tracefree.

Degrees of Freedom

Kinematical variables

- Lapse α : 1 degree of freedom
- Shift β^i : 3 degrees of freedom

Initial-data variables

- Metric γ_{ij} : 6 degrees of freedom
- Extrinsic curvature K_{ij} : 6 degrees of freedom

Decomposition of initial-data variables

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \left\{ \begin{array}{l} \psi : 1 \text{ constrained DOF} \\ \tilde{\gamma}_{ij} : \left\{ \begin{array}{l} 3 \text{ spatial gauge DOF} \\ 2 \text{ dynamical DOF} \end{array} \right\} \end{array} \right\} \text{Freely Specifiable}$$

$$K^{ij} = \psi^{-10} \left[\frac{1}{\tilde{\sigma}} (\tilde{\mathbb{L}}V)^{ij} + \tilde{M}^{ij} \right] + \frac{1}{3} \psi^{-4} \tilde{\gamma}^{ij} K \left\{ \begin{array}{l} V^i : \left\{ \begin{array}{l} (\tilde{\mathbb{L}}V)^{ij} \equiv \tilde{\nabla}^i V^j + \tilde{\nabla}^j V^i - \frac{1}{3} \tilde{\gamma}^{ij} \tilde{\nabla}_k V^k \\ 3 \text{ constrained DOF} \end{array} \right\} \\ \tilde{M}^{ij} : \left\{ \begin{array}{l} \tilde{\nabla}_j \tilde{M}^{ij} = \tilde{\nabla}_j \left(\frac{1}{\tilde{\sigma}} (\tilde{\mathbb{L}}X)^{ij} \right) \\ 2 \text{ dynamical DOF} \end{array} \right\} \\ K : 1 \text{ temporal gauge DOF} \\ \tilde{\sigma} : \text{Defn. of TT decomp.} \end{array} \right\} \text{Freely Specifiable}$$

“Traditional” Black-Hole Data

Conformal flatness and maximal slicing

$$\left. \begin{aligned} \tilde{\gamma}_{ij} &= f_{ij} \text{ (flat)} \\ \tilde{M}^{ij} &= 0 \\ K &= 0 \\ \tilde{\sigma} &= 1 \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} \tilde{\nabla}_i (\tilde{\mathbb{L}}V)^{ij} &= 0 \Rightarrow \\ \tilde{\nabla}^2 \psi + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} &= 0 \end{aligned} \right.$$

Bowen-York solution [4]
Analytic solutions for \tilde{A}^{ij}

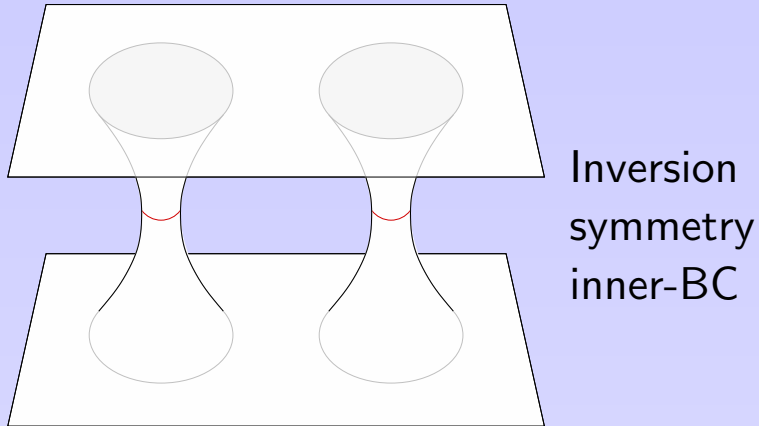
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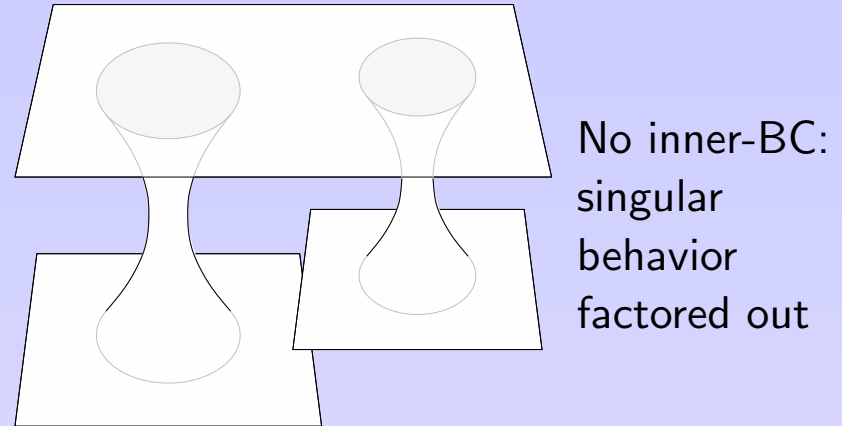
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Three general solution schemes

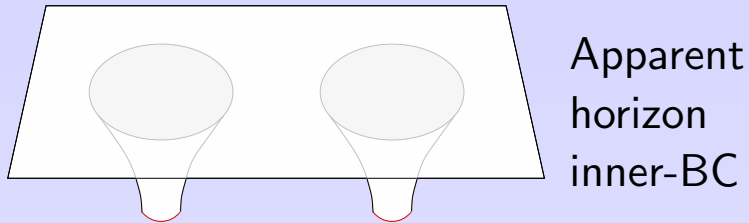
Conformal Imaging-[7]



Puncture Method-[5]



Apparent Horizon BC-[14]



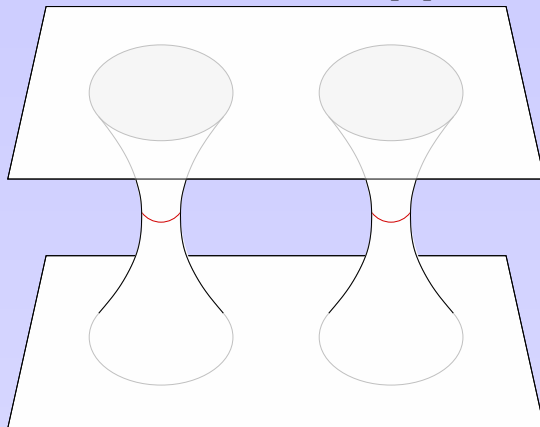
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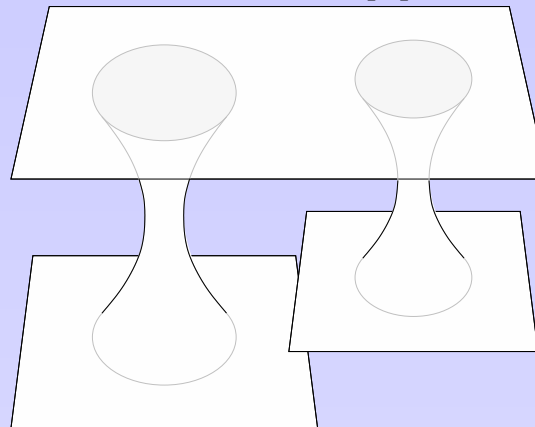
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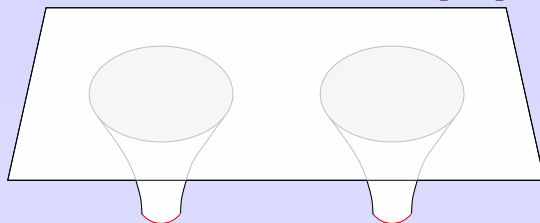
Inversion
symmetry
inner-BC

Puncture Method-[5]



No inner-BC:
singular
behavior
factored out

Apparent Horizon BC-[14]



Apparent
horizon
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All methods can produce very general configurations of multiple black holes, but are fundamentally limited by choices for $\tilde{\gamma}_{ij}$ and Bowen-York \tilde{A}^{ij} .

“Better” Black-Hole Data

What is wrong with “traditional” BH initial data?

- Results disagree with PN predictions for black holes in quasi-circular orbits.
- There is no control of the initial “wave” content.
- Spinning holes are not represented well.

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How do we construct improved BH initial data?

We must carefully choose the

- initial dynamical degrees of freedom [in $\tilde{\gamma}_{ij}$ and \tilde{M}^{ij}]
- initial temporal and spatial gauge degrees of freedom [in $\tilde{\gamma}_{ij}$ and K]
- boundary conditions on the constrained degrees of freedom [in ψ and V^i]
- TT decomposition weight function [$\tilde{\sigma}$]

so as to conform to the desired physical content of the initial data.

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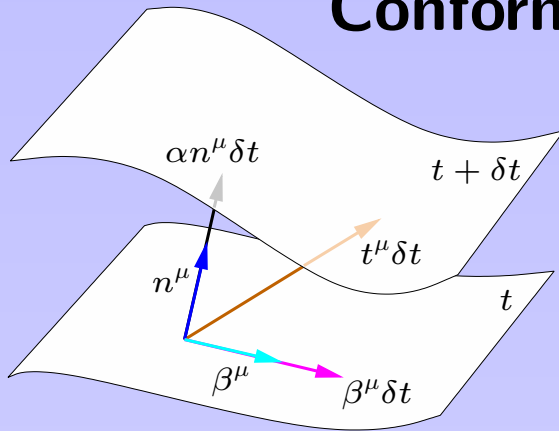
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- For black holes in quasi-circular orbits, we can use the principle of *quasi-equilibrium* to guide our choices.
- *Quasi-equilibrium* is a *dynamical* concept and we can simplify our task by choosing a decomposition of the initial-data variables that has connections to dynamics.

Conformal Thin-Sandwich Decomposition



$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$$

$$K^{ij} = \frac{\psi^{-10}}{2\tilde{\alpha}} \left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right] + \frac{1}{3}\psi^{-4}\tilde{\gamma}^{ij}K$$

Hamiltonian Const. $\tilde{\nabla}^2\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{12}\psi^5K^2 + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = -2\pi\psi^5\rho$

Momentum Const. $\tilde{\nabla}_j(\tilde{\mathbb{L}}\beta)^{ij} - (\tilde{\mathbb{L}}\beta)^{ij}\tilde{\nabla}_j\tilde{\alpha} = \frac{4}{3}\tilde{\alpha}\psi^6\tilde{\nabla}^iK + \tilde{\alpha}\tilde{\nabla}_j\left(\frac{1}{\tilde{\alpha}}\tilde{u}^{ij}\right) + 16\pi\tilde{\alpha}\psi^{10}j^i$

Const. Tr(K) eqn. $\tilde{\nabla}^2(\psi^7\tilde{\alpha}) - (\psi^7\tilde{\alpha})\left[\frac{1}{8}\psi\tilde{R} + \frac{5}{12}\psi^5K^2 + \frac{7}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} - \psi^5\beta^i\tilde{\nabla}_iK\right]$
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Constrained vars : ψ and β^i

Freely specified : $\tilde{\gamma}_{ij}$, \tilde{u}^{ij} , K , and ∂_tK

\tilde{u}^{ij} and β^i have a simple physical interpretation, unlike \tilde{M}^{ij} and V^i .

$$\text{Quasi-equilibrium} \Rightarrow \begin{cases} \tilde{u}^{ij} = 0 \\ \partial_tK = 0 \end{cases}$$

Equations of Quasi-Equilibrium

$$\left. \begin{array}{l} \text{Ham. \& Mom. const.} \\ \text{eqns., \& Const Tr}(K) \\ \text{eqn. from Conf. TS} \\ + \tilde{u}^{ij} = \partial_t K = 0 \end{array} \right\} \Rightarrow \text{Eqns. of} \\ \text{Quasi-Equilibrium}$$

With $\tilde{\gamma}_{ij} = f_{ij}$ and $K = 0$, these equations have been widely used to construct binary neutron star initial data [2, 12, 3, 15].

Binary neutron star initial data require:

- boundary conditions at infinity compatible with asymptotic flatness and corotation.

$$\psi|_{r \rightarrow \infty} = 1 \quad \beta^i|_{r \rightarrow \infty} = \Omega \left(\frac{\partial}{\partial \phi} \right)^i \quad \alpha|_{r \rightarrow \infty} = 1$$

- compatible solution of the equations of hydrostatic equilibrium. ($\Rightarrow \Omega$)

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Binary black hole initial data require:

- a means for choosing the angular velocity of the orbit Ω .

★ *with excision*, inner boundary conditions are needed for ψ , β^i , and $\tilde{\alpha}$.

Gourgoulhon, Grandclément, & Bonazzola[10, 11]: Black-hole binaries with $\tilde{\gamma}_{ij} = f_{ij}$ & $K = 0$, “inversion-symmetry”, and “Killing-horizon” conditions on the excision boundaries.

“Solutions” require *constraint violating* regularity condition imposed on inner boundaries!

Constructing Regular Binary Black Hole QE ID

Why does the GGB approach have problems?

- Inversion-symmetry demands $\tilde{\alpha} = 0$ on the inner boundaries.

Inversion-symmetry BC's are not compatible

- with $(\tilde{\mathbb{L}}\beta)^{ij} = 0$ **and** $\tilde{u}^{ij} = 0$, unless ∂_t is a true Killing vector. [conjecture]

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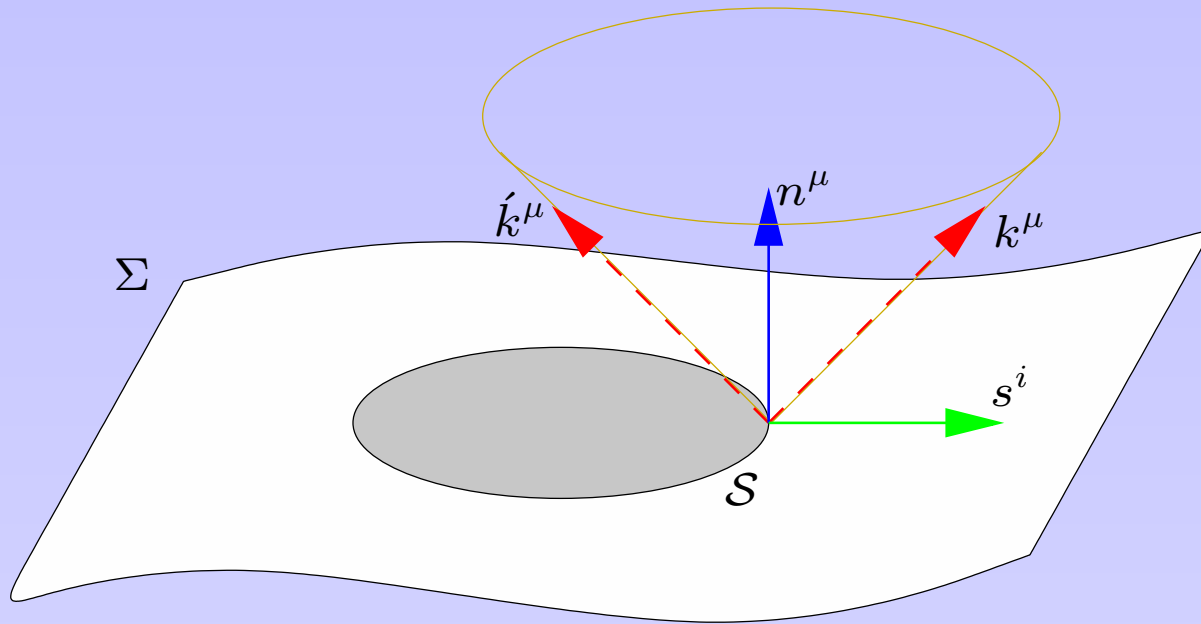
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Approach

- Demand that the excision (*inner*) boundary be an *apparent horizon*.
- Demand that the apparent horizon be in quasi-equilibrium.

The Inner Boundary



$$s_i \equiv \frac{\bar{\nabla}_i \tau}{|\bar{\nabla} \tau|}$$

$$h_{ij} \equiv \gamma_{ij} - s_i s_j$$

$$k^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu + s^\mu)$$

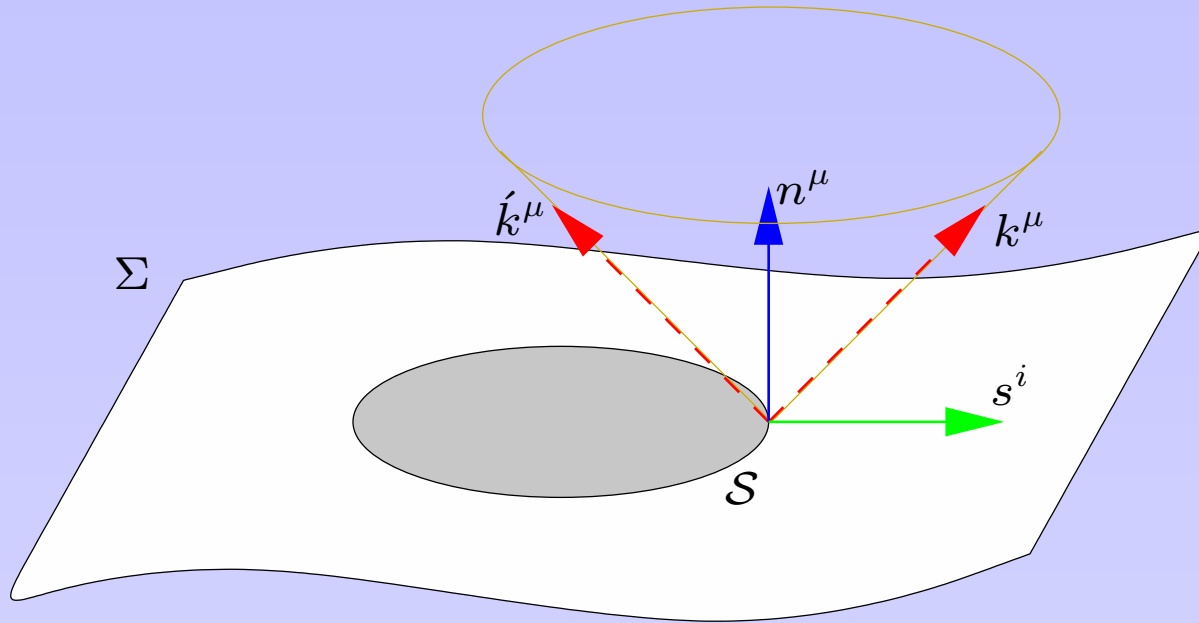
$$\hat{k}^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu - s^\mu)$$

Extrinsic curvature of S embedded in
spacetime

$$\Sigma_{\mu\nu} \equiv \frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_k g_{\alpha\beta}$$

$$\hat{\Sigma}_{\mu\nu} \equiv \frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_{\hat{k}} g_{\alpha\beta}$$

The Inner Boundary



$$s_i \equiv \frac{\bar{\nabla}_i \tau}{|\bar{\nabla} \tau|}$$

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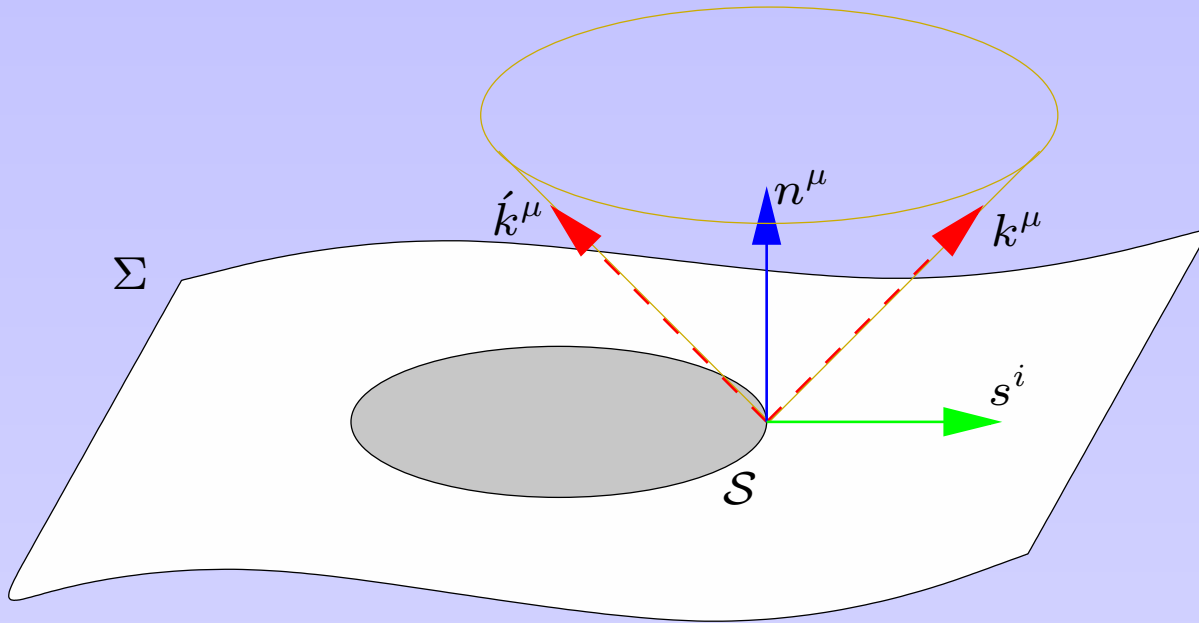
Projections of K_{ij} onto S

$$J_{ij} \equiv h_i^k h_j^\ell K_{kl}$$

$$J_i \equiv h_i^k s^\ell K_{kl}$$

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Expansion of null rays

$$\theta \equiv h^{ij} \Sigma_{ij} = \frac{1}{\sqrt{2}} (H - J)$$

$$\hat{\theta} \equiv h^{ij} \hat{\Sigma}_{ij} = -\frac{1}{\sqrt{2}} (H + J)$$

Shear of null rays

$$\sigma_{ij} \equiv \Sigma_{ij} - \frac{1}{2} h_{ij} \theta$$

$$\hat{\sigma}_{ij} \equiv \hat{\Sigma}_{ij} - \frac{1}{2} h_{ij} \hat{\theta}$$

AH and QE Conditions on the Inner Boundary

The quasi-equilibrium inner boundary conditions start with the following assumptions:

1. The inner boundary \mathcal{S} is a (MOTS):
marginally outer-trapped surface

$$\rightarrow \theta = 0$$

2. The inner boundary \mathcal{S} doesn't move:

$$\rightarrow \mathcal{L}_\zeta \tau = 0 \text{ and } D_i \mathcal{L}_\zeta \tau \equiv h_i^j \bar{\nabla}_j \mathcal{L}_\zeta \tau = 0$$

$$t^\mu = \alpha n^\mu + \beta^\mu \quad \zeta^\mu \equiv \alpha n^\mu + \beta_\perp s^\mu$$

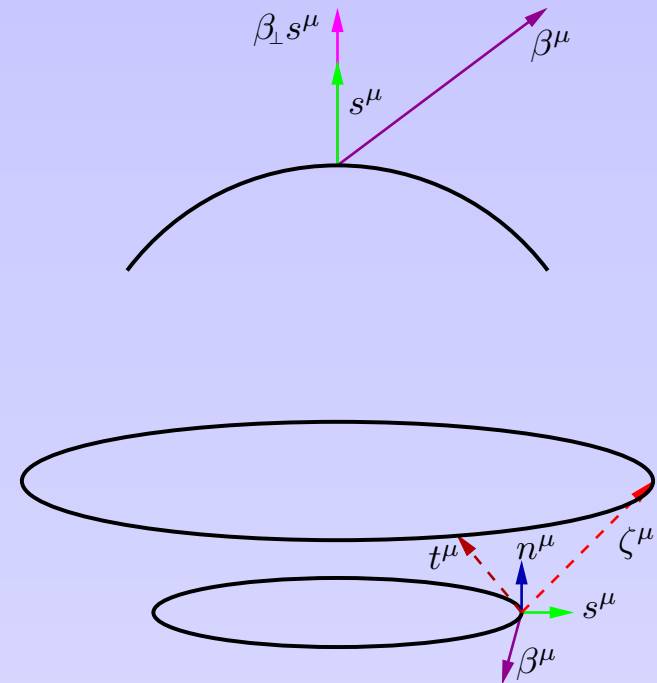
$$\beta_\perp \equiv \beta^i s_i$$

3. The inner boundary \mathcal{S} remains a MOTS:

$$\rightarrow \mathcal{L}_\zeta \theta = 0$$

4. The horizons are in quasi-equilibrium:

$$\rightarrow \sigma_{ij} = 0 \text{ and no matter is on } \mathcal{S}$$



AH/Quasi-Equilibrium Boundary Conditions

$$\theta = \frac{\psi^{-2}}{\sqrt{2}} \left[\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j + 4\tilde{s}^k \tilde{\nabla}_k \ln \psi - \psi^2 J \right]$$

$$\mathcal{L}_\zeta \theta = -\frac{1}{\sqrt{2}} \left[\theta(\theta + \frac{1}{2}\dot{\theta} + \frac{1}{\sqrt{2}}K) + \mathcal{E} \right] (\beta_\perp + \alpha)$$

$$-\frac{1}{\sqrt{2}} \left[\theta(\frac{1}{2}\theta - \frac{1}{2}\dot{\theta} + \frac{1}{\sqrt{2}}K) + \mathcal{D} + 8\pi T_{\mu\nu} k^\mu \acute{k}^\nu \right] (\beta_\perp - \alpha)$$

$$+ \theta s^i \bar{\nabla}_i \alpha$$

$$\mathcal{D} \equiv h^{ij} (D_i + J_i)(D_j + J_j) - \frac{1}{2} R$$

$$\mathcal{E} \equiv \sigma_{ij} \sigma^{ij} + 8\pi T_{\mu\nu} k^\mu k^\nu$$

$$\sigma_{ij} = \frac{1}{\sqrt{2}} (H_{ij} - \frac{1}{2} h_{ij} H) \left(1 - \frac{\beta_\perp}{\alpha} \right)$$

$$-\frac{1}{\sqrt{2}} \frac{\psi^4}{\alpha} \left\{ \tilde{D}_{(i} \beta_{\parallel j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \beta_{\parallel}^k - \frac{1}{2} [\tilde{h}_{ik} \tilde{h}_{jl} \tilde{u}^{kl} - \frac{1}{2} \tilde{h}_{ij} \tilde{h}_{kl} \tilde{u}^{kl}] \right\}$$

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$$\tilde{s}^k \tilde{\nabla}_k \ln \psi = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J)$$

$$\beta^i = \alpha \psi^{-2} \tilde{s}^i + \beta_{\parallel}^i$$

$$0 = \tilde{D}_{(i} \beta_{\parallel j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \beta_{\parallel}^k$$

Defining the Spin of the Black Hole

The spin parameters β_{\parallel}^i can be defined by demanding that the time vector associated with quasi-equilibrium in the corotating frame must be null, forming the null generators of the horizon.

$$k^{\mu} \propto (n^{\mu} + s^{\mu}) \implies k^{\mu} = [1, \alpha s^i - \beta^i]$$

This vector k^{μ} is null for any choice of α & β^i .

In the frame where a black hole is not spinning, the null time vector has components $t^{\mu} = [1, \vec{0}]$.

Corotating Holes

Corotating holes are at rest in the corotating frame, where we must pose boundary conditions. So,

$$k^{\mu} = [1, \alpha s^i - \beta^i] = [1, \vec{0}]$$

Thus we find

$$\beta^i = \alpha s^i \implies \beta_{\parallel}^i = 0$$

Irrotational Holes

Irrotational holes are at rest in the inertial frame. With the time vectors in the inertial and corotating frames related by

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} + \Omega \frac{\partial}{\partial \phi}$$

$$k^{\mu} = [1, \alpha s^i - \beta^i] = [1, -\Omega(\partial/\partial\phi)^i]$$

Thus we find

$$\beta^i = \alpha s^i + \Omega \left(\frac{\partial}{\partial \phi} \right)^i \implies \beta_{\parallel}^i = \Omega \xi^i$$

$$\xi^i \approx \left(\frac{\partial}{\partial \phi} \right)^i \quad \& \quad \tilde{D}_{(i} \xi_{j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \xi^k = 0$$

The Orbital Angular Velocity

- For a given choice of the Lapse BC, $\tilde{\gamma}_{ij}$ and K , we are still left with a family of solutions parameterized by the orbital angular velocity Ω .
- Except for the case of a single spinning black hole, it is not reasonable to expect more than one value of Ω to correspond to a system in quasi-equilibrium.

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GGB[10, 11] have suggested a way to pick the quasi-equilibrium value of Ω :

Ω is chosen as the value for which the ADM energy E_{ADM} equals the Komar mass M_{K} .

Komar mass	$M_{\text{K}} = \frac{1}{4\pi} \oint_{\infty} \gamma^{ij} (\bar{\nabla}_i \alpha - \beta^k K_{ik}) d^2 S_j$	Acceptable definition of the mass <i>only for stationary spacetimes.</i>
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ADM energy	$E_{\text{ADM}} = \frac{1}{16\pi} \oint_{\infty} \gamma^{ij} \bar{\nabla}_k (\mathcal{G}_i^k - \delta_i^k \mathcal{G}) d^2 S_j$	Acceptable definition of the mass <i>for arbitrary spacetimes.</i> $\mathcal{G}_{ij} \equiv \gamma_{ij} - f_{ij}$
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The Lapse BC & QE

So far, nothing has fixed a boundary condition on the lapse α . One possibility[9] is to recall that $\theta\acute{\theta}$ is a Lorentz invariant and so to consider $\mathcal{L}_\zeta\theta\acute{\theta} = 0$ as a quasi-equilibrium condition.

$$\mathcal{L}_\zeta\theta\acute{\theta} = 0 \quad \Rightarrow \quad J\tilde{s}^i\tilde{\nabla}_i\alpha = -\psi^2(J^2 - JK + \tilde{\mathcal{D}})\alpha$$

$$\tilde{\mathcal{D}} \equiv \psi^{-4}[\tilde{h}^{ij}(\tilde{D}_i - J_i)(\tilde{D}_j - J_j) - \frac{1}{2}\tilde{R} + 2\tilde{D}^2 \ln \psi]$$

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This conditions is satisfied for stationary solutions, but seems to be degenerate with the other QE boundary conditions. To see this, note that the stationary maximal slicings of Schwarzschild form a 1-parameter family:

$$\begin{aligned} ds^2 &= \frac{dR^2}{1 - \frac{2M}{R} + \frac{C^2}{R^4}} + R^2 d^2\Omega \\ \alpha &= \sqrt{1 - \frac{2M}{R} + \frac{C^2}{R^4}} \\ \beta^R &= \frac{C}{R^2} \sqrt{1 - \frac{2M}{R} + \frac{C^2}{R^4}} \end{aligned} \quad K_j^i = \frac{C}{R^3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

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$$\alpha|_S = \frac{C}{4M^2}$$

Summary of QE Formalism

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \gamma^{ij} K \quad \tilde{A}^{ij} = \frac{\psi^6}{2\alpha} (\tilde{\mathbb{L}}\beta)^{ij} \quad \partial_t \tilde{\gamma}_{ij} = 0$$

$$\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0$$

$$\tilde{\nabla}_j (\tilde{\mathbb{L}}\beta)^{ij} - (\tilde{\mathbb{L}}\beta)^{ij} \tilde{\nabla}_j \ln \alpha \psi^{-6} = \frac{4}{3} \alpha \tilde{\nabla}^i K$$

$$\tilde{\nabla}^2 (\alpha \psi) - (\alpha \psi) \left[\frac{1}{8} \tilde{R} + \frac{5}{12} \psi^4 K^2 + \frac{7}{8} \psi^{-8} A_{ij} A^{ij} \right] = \psi^5 \beta^i \tilde{\nabla}_i K \quad \partial_t K = 0$$

$$\tilde{s}^k \tilde{\nabla}_k \ln \psi|_S = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J)|_S \quad \theta = 0$$

$$\beta^i|_S = \begin{cases} \alpha \psi^{-2} \tilde{s}^i|_S & \text{corotation} \\ \alpha \psi^{-2} \tilde{s}^i|_S + \Omega \xi^i|_S & \text{irrotation} \end{cases}$$

$$\mathcal{L}_\zeta \theta = 0$$

$$\sigma_{ij} = 0$$

$\alpha|_S =$ unspecified by QE

$$\psi|_{r \rightarrow \infty} = 1$$

$$\beta^i|_{r \rightarrow \infty} = \Omega \left(\frac{\partial}{\partial \phi} \right)^i$$

$$\alpha|_{r \rightarrow \infty} = 1$$

The only remaining freedom in the system is the choice of the lapse boundary condition, the initial spatial and temporal gauge, and the initial dynamical (“wave”) content found in

$\alpha|_S$, $\tilde{\gamma}_{ij}$ and K .

Results

Corotation

$\tilde{\gamma}_{ij} = f_{ij}$: Maximal Slicing:

- $\frac{\partial(\alpha\psi)}{\partial r} = 0$
- $\alpha\psi = \frac{1}{2}$
- $\frac{\partial(\alpha\psi)}{\partial r} = \frac{\alpha\psi}{2r}$

$\tilde{\gamma}_{ij} = f_{ij}$: Eddington-Finkelstein Slicing:

- $\frac{\partial(\alpha\psi)}{\partial r} = 0$
- $\alpha\psi = \frac{1}{2}$
- $\frac{\partial(\alpha\psi)}{\partial r} = \alpha\psi$

Compared with

- Effective-One-Body PN[8]
- Inversion-Symmetric HKV[11]

Irrotation

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Compared with

- Effective-One-Body PN[8]
- Conformal Imaging[6]
- Puncture Method[1]

Constructing Evolutionary Sequences

Given quasi-equilibrium data for binary black holes at various separations, how do we connect them together to form an evolutionary sequence?

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- Let $e(s)$, $j(s)$, and $\omega(s)$ denote the numerical solutions along a sequence parameterized by some measure of the separation s .
- Let $\chi(s)$ define a mass scaling along the same sequence.

$$E_{\text{ADM}}(s) \equiv \chi(s)m(s)$$

- Define $J(s) \equiv \chi^2(s)j(s)$
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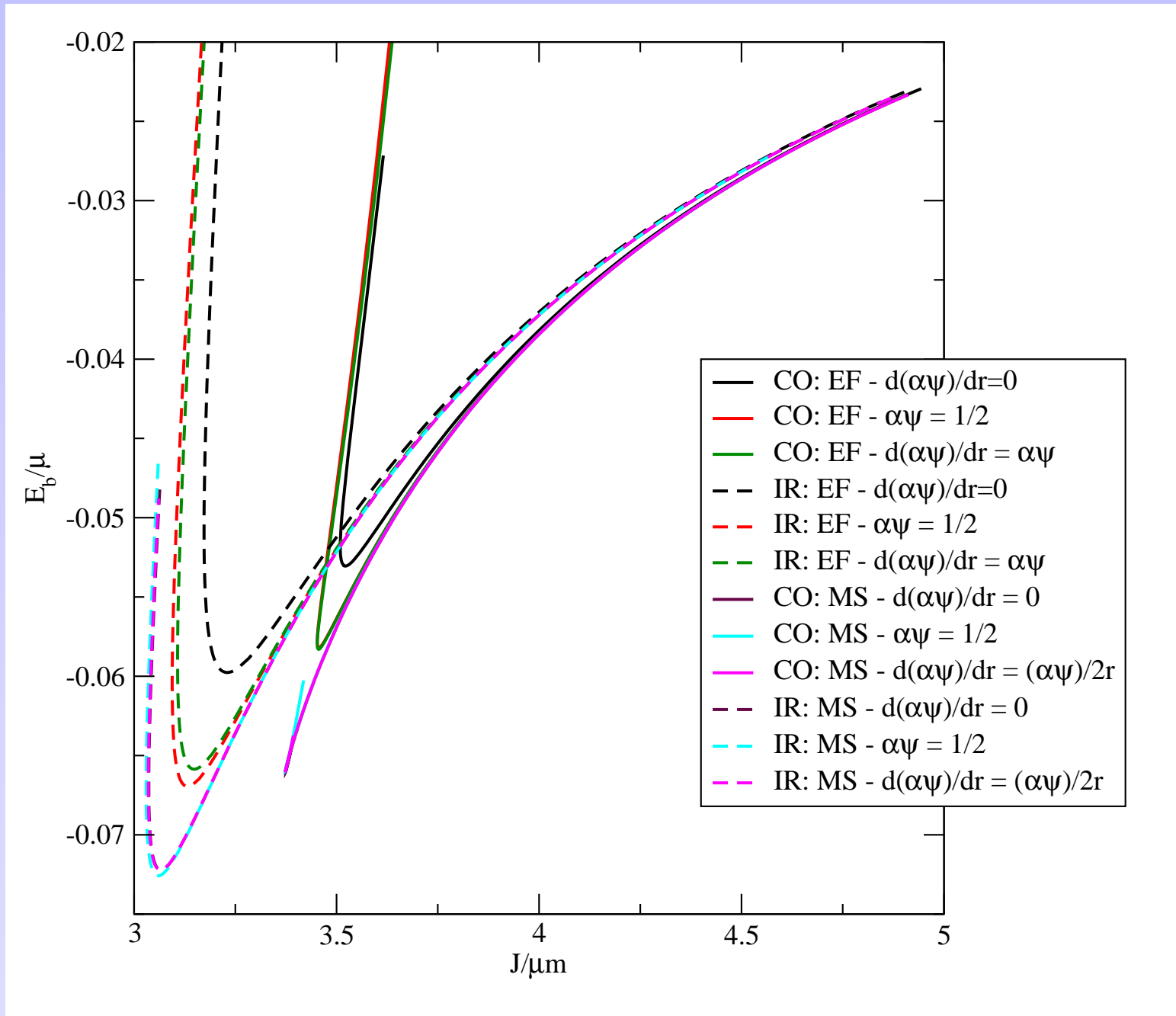
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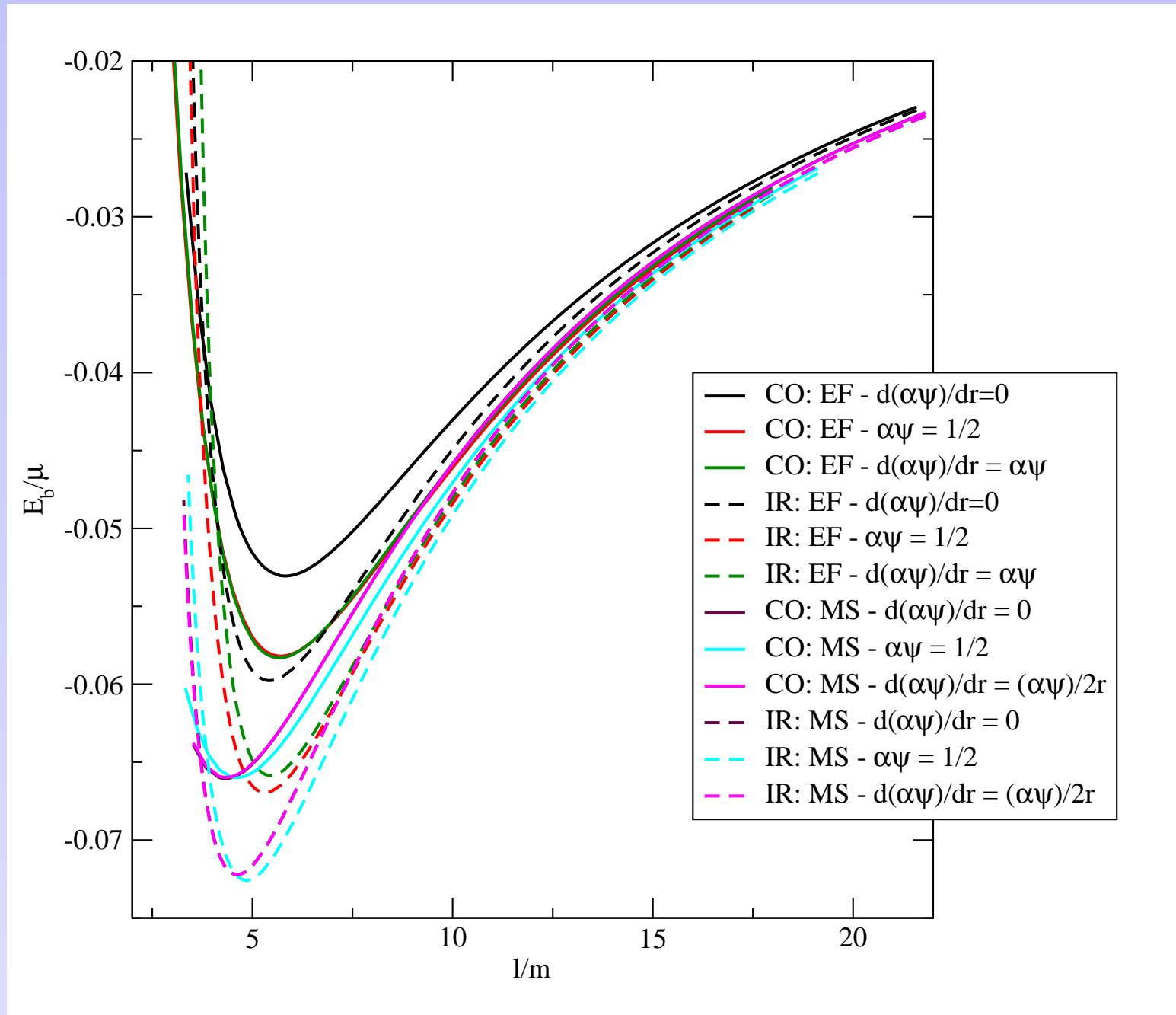
⇒ Choose the mass scale to preserve $dE_{\text{ADM}} = \Omega dJ$.

- Integrating from s_1 to s_2 gives: $\chi(s_2) = \chi(s_1) \exp \left\{ - \int_{s_1}^{s_2} \frac{m' - \omega j'}{m - 2\omega j} ds \right\}$

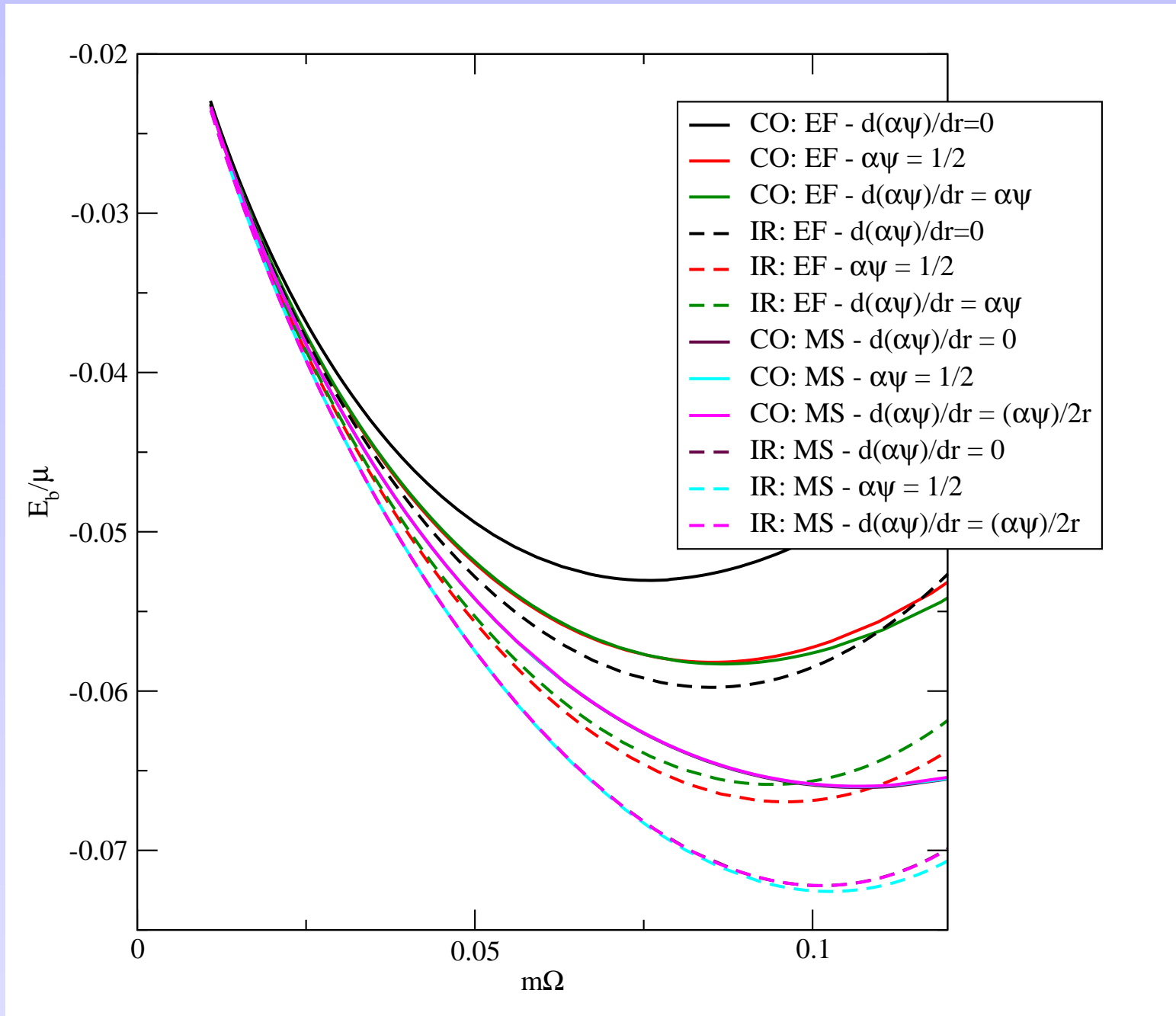
Results: E_b/μ vs $J/\mu m$



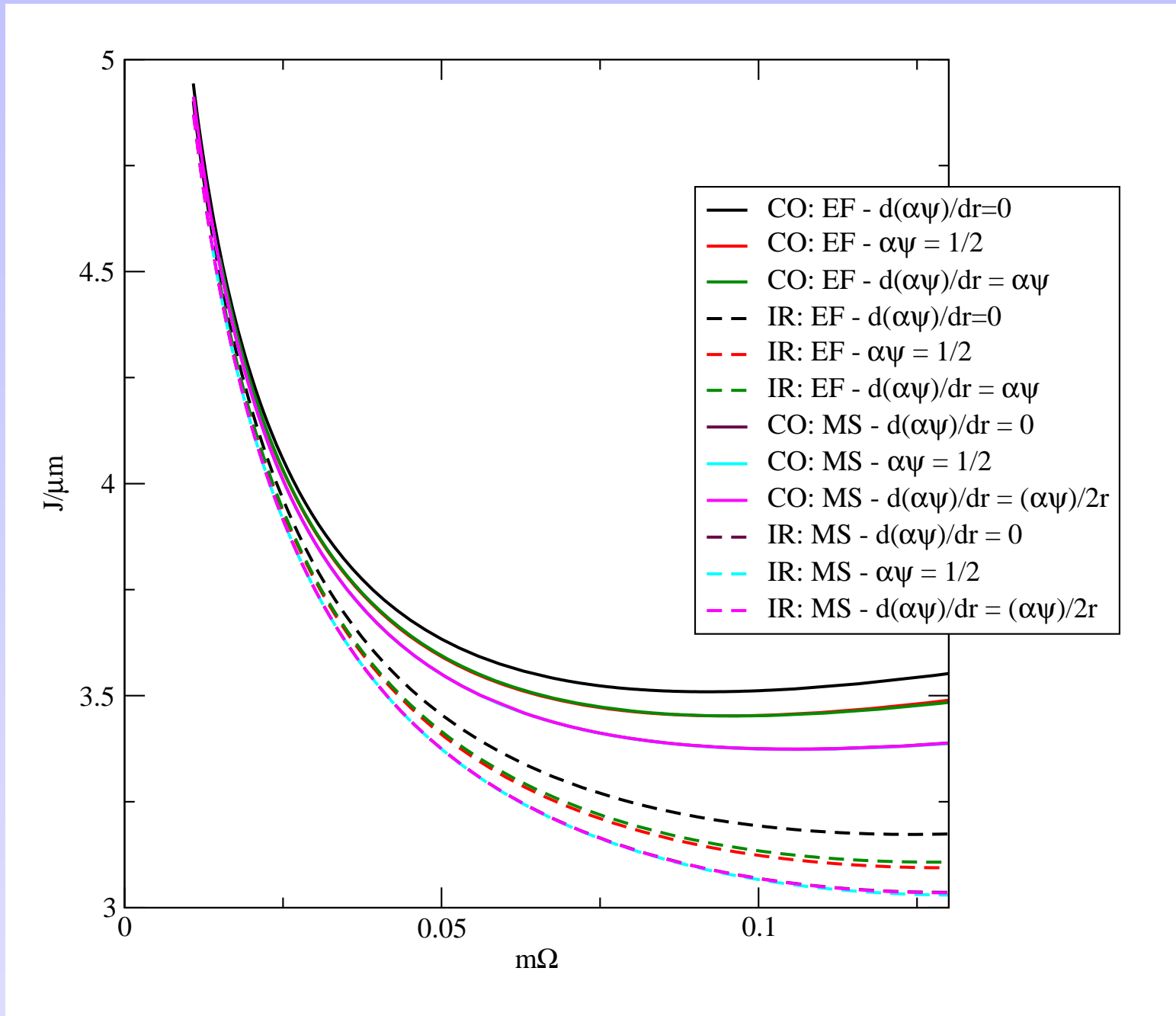
Results: E_b/μ vs ℓ/m



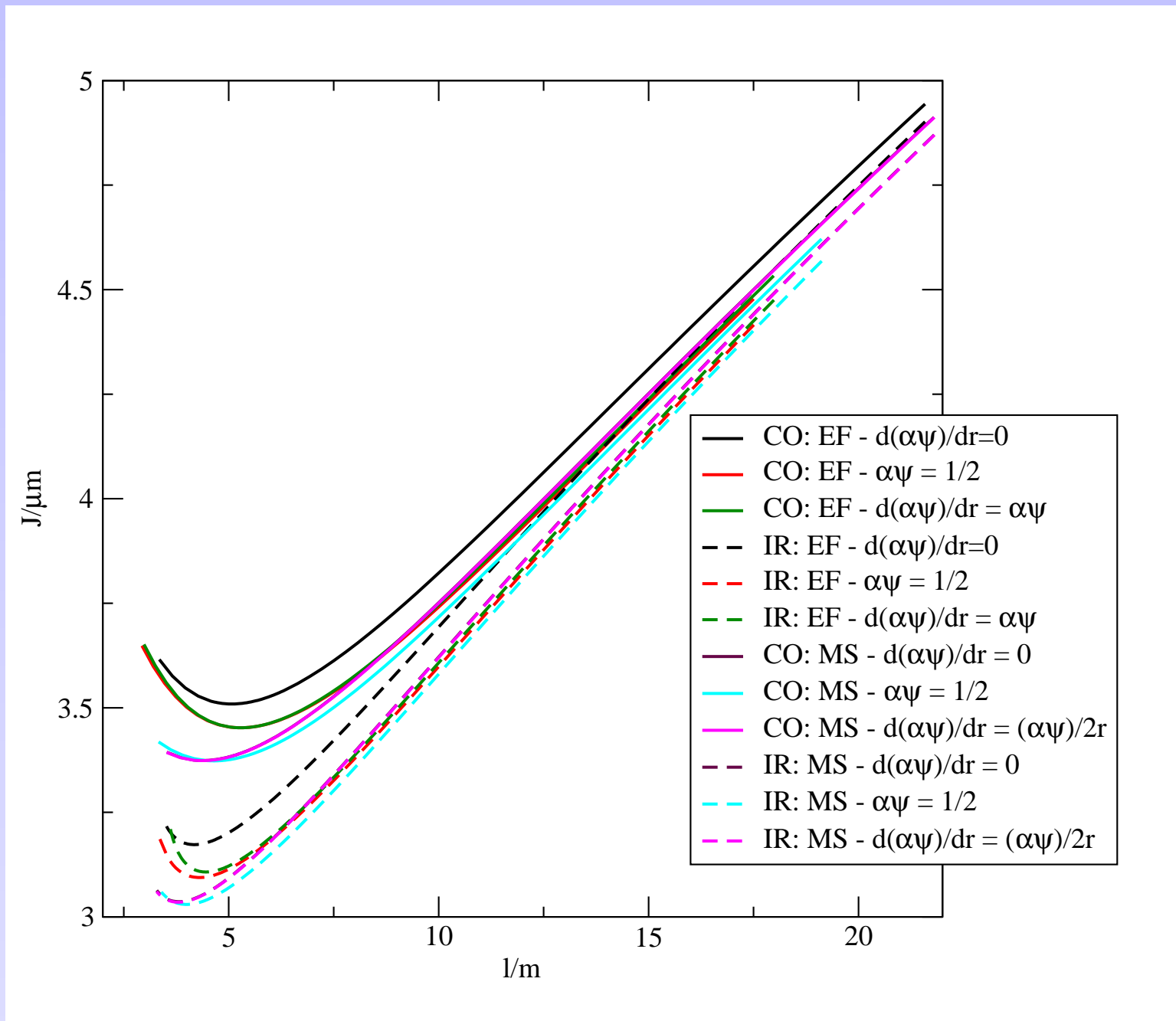
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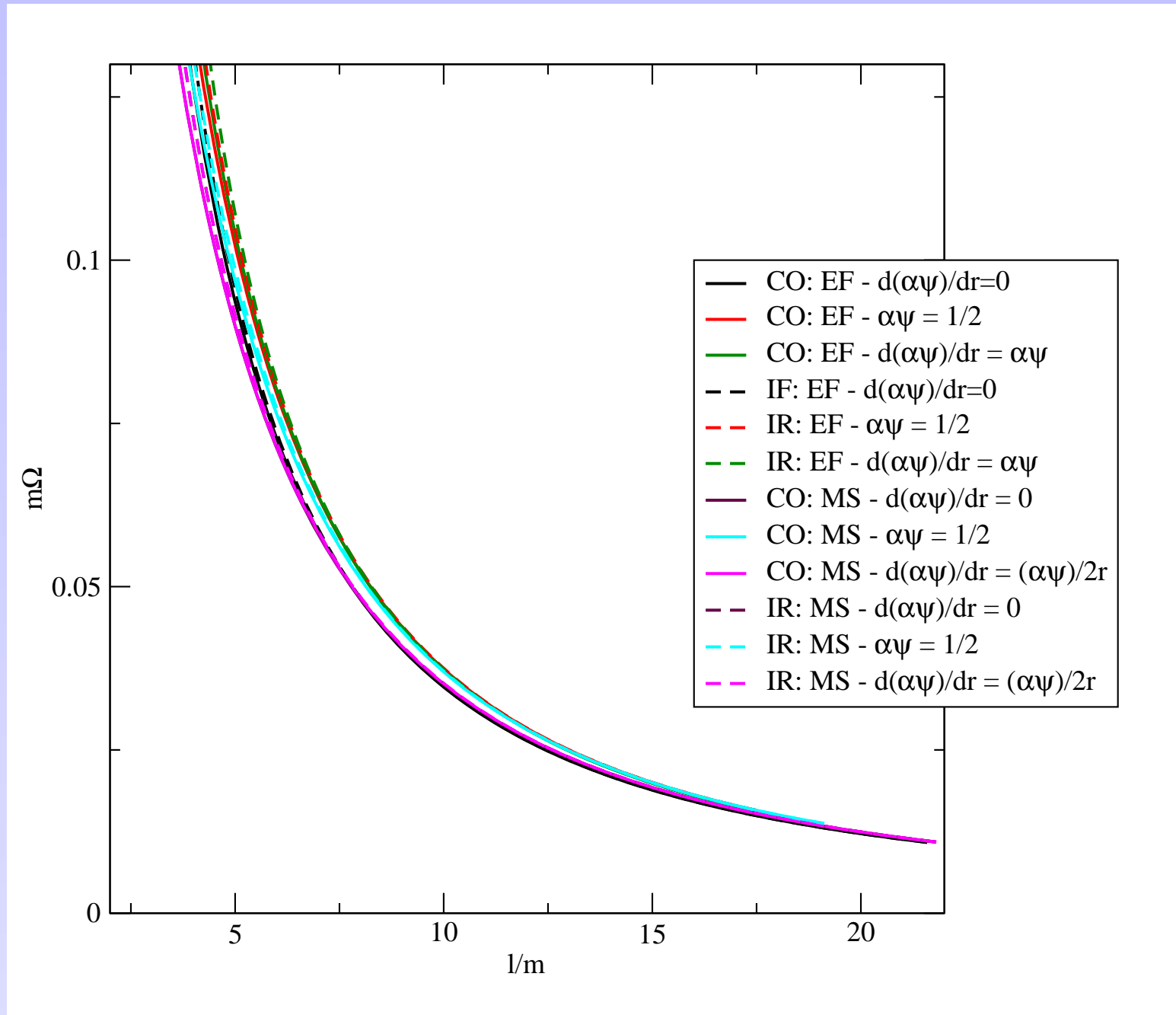
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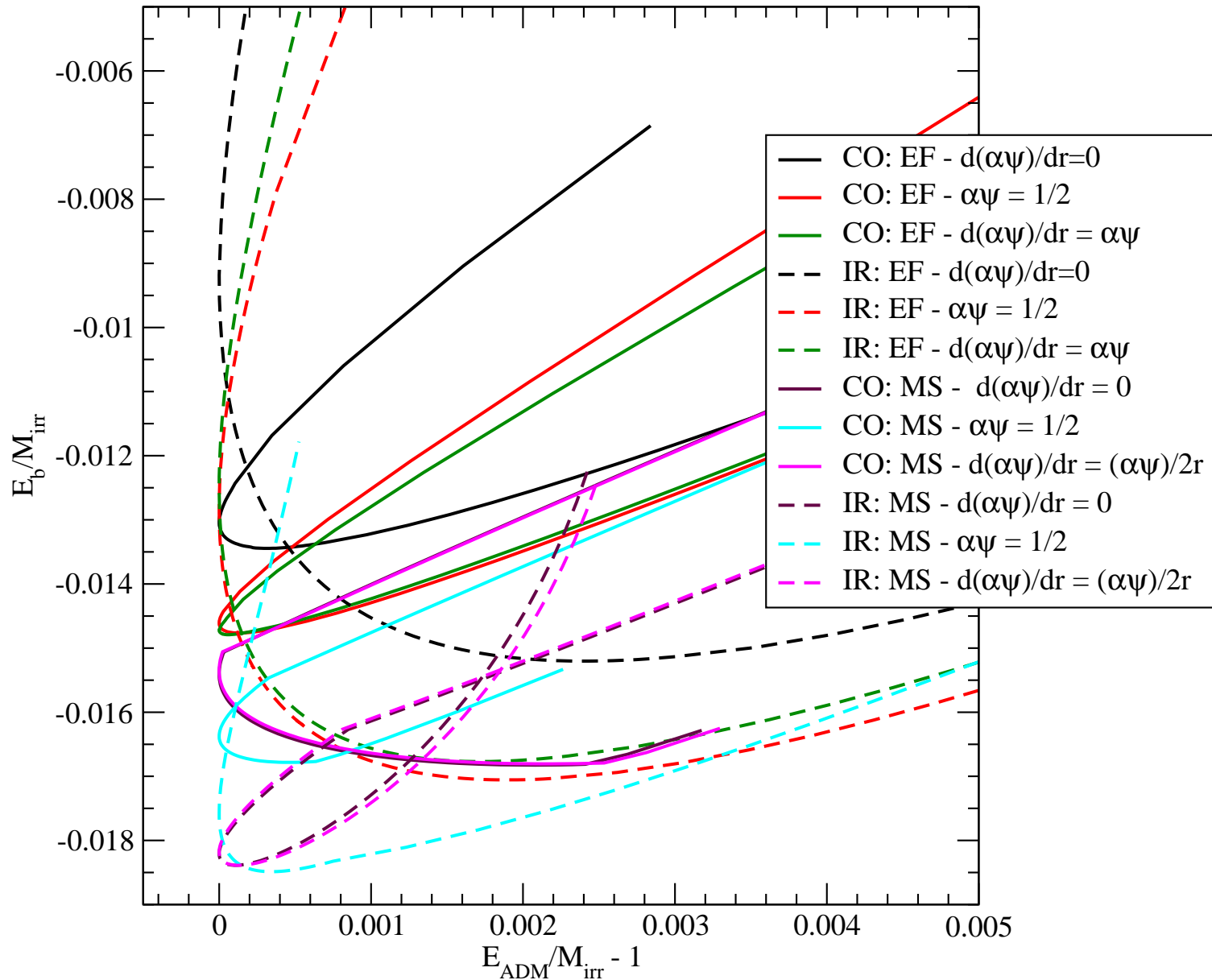
Results: $J/\mu m$ vs ℓ/m



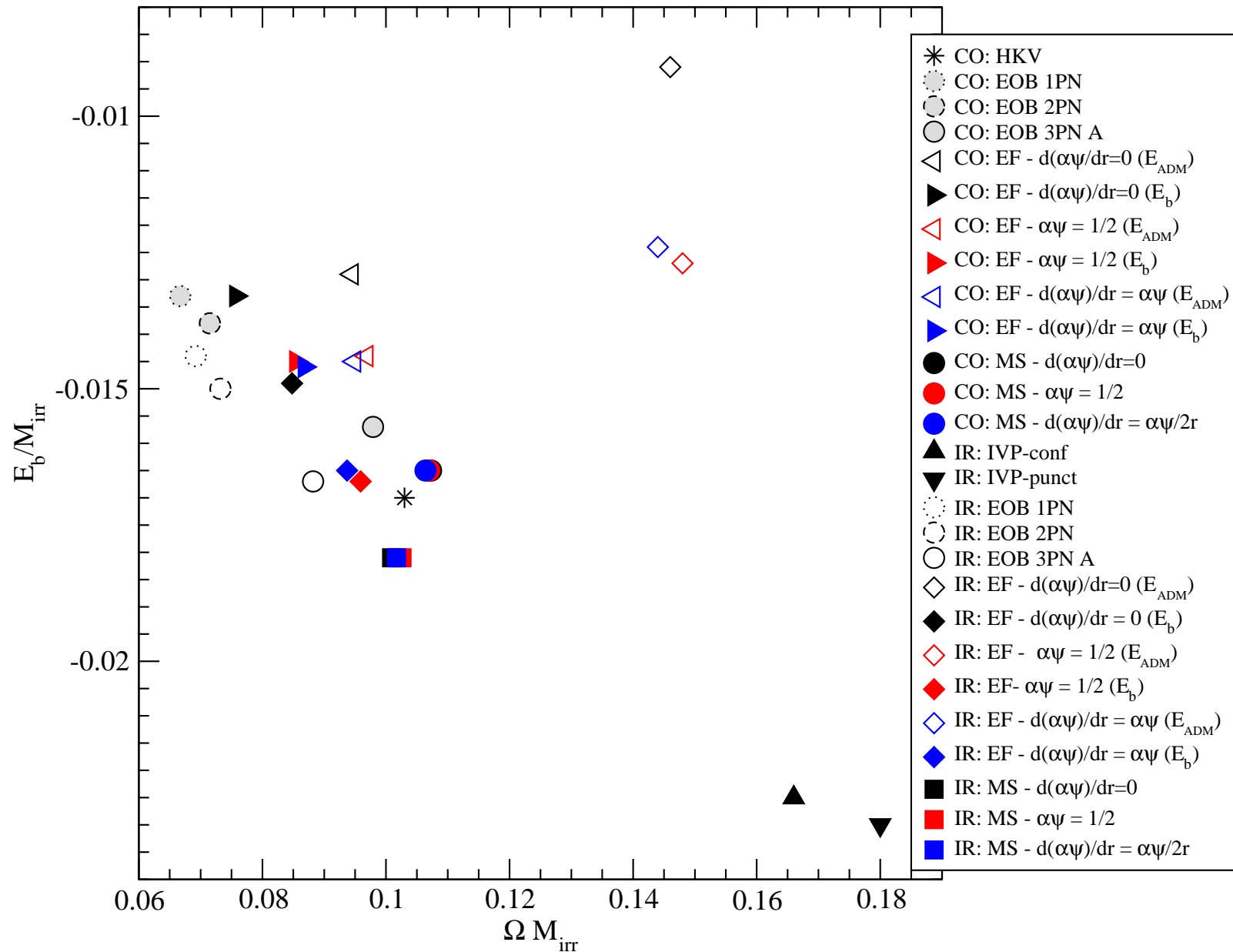
Results: $m\Omega$ vs ℓ/m



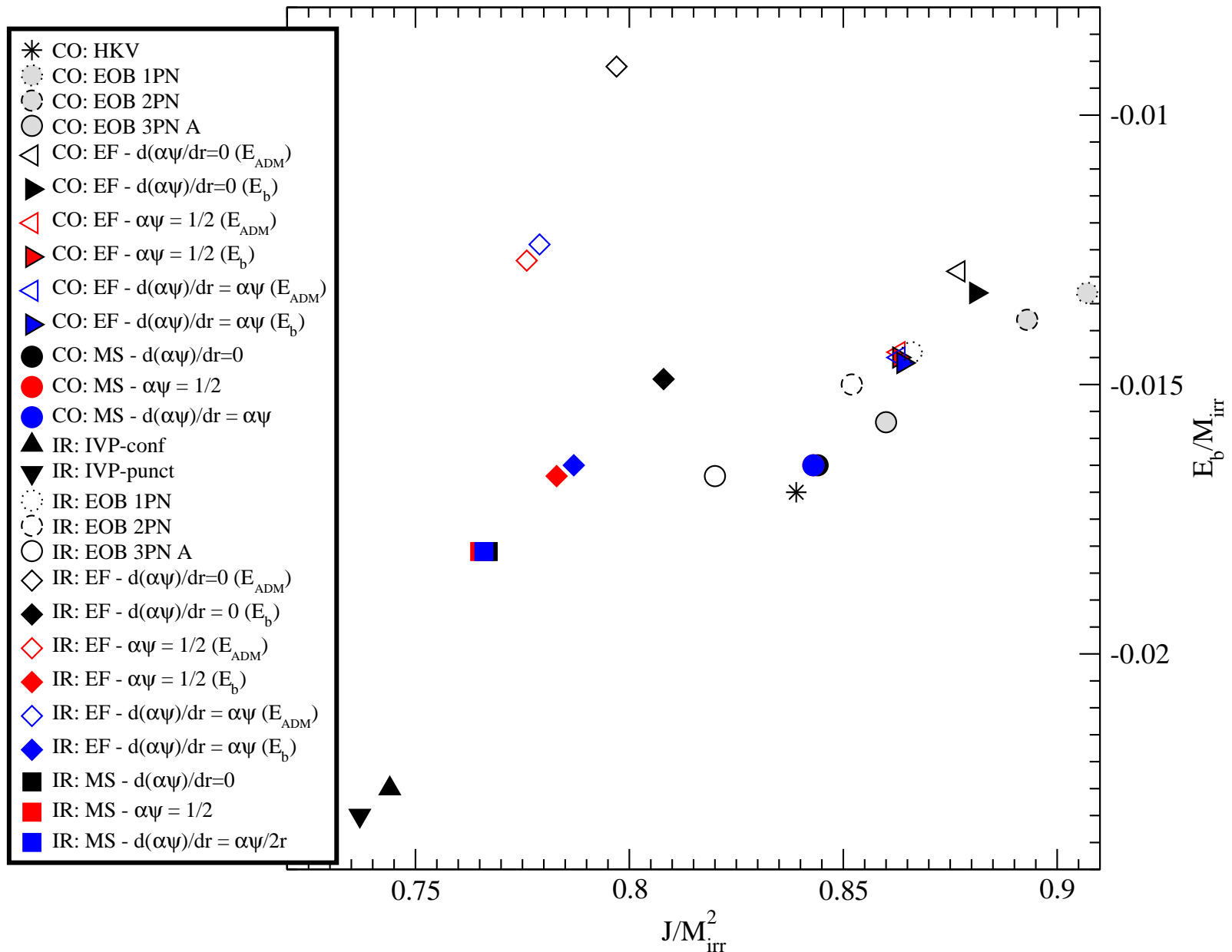
Results: ISCO — E_b/M_{irr} vs $E_{\text{ADM}}/M_{\text{irr}} - 1$



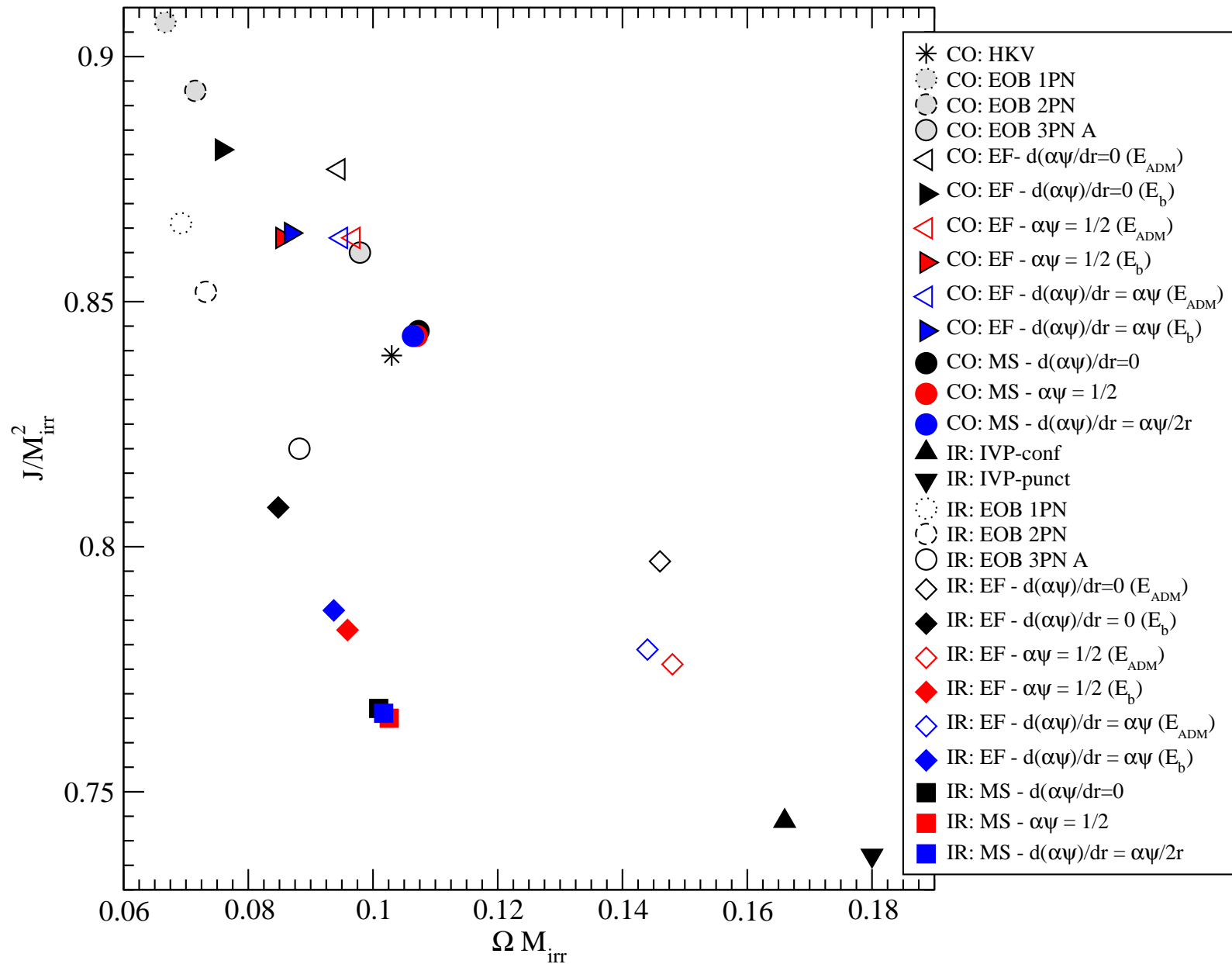
Results: ISCO — E_b/M_{irr} vs ΩM_{irr}



Results: ISCO — E_b/M_{irr} vs J/M_{irr}^2



Results: ISCO — J/M_{irr}^2 vs ΩM_{irr}



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