Generating Binary Black Hole Initial Data

Gregory B. Cook

Wake Forest University

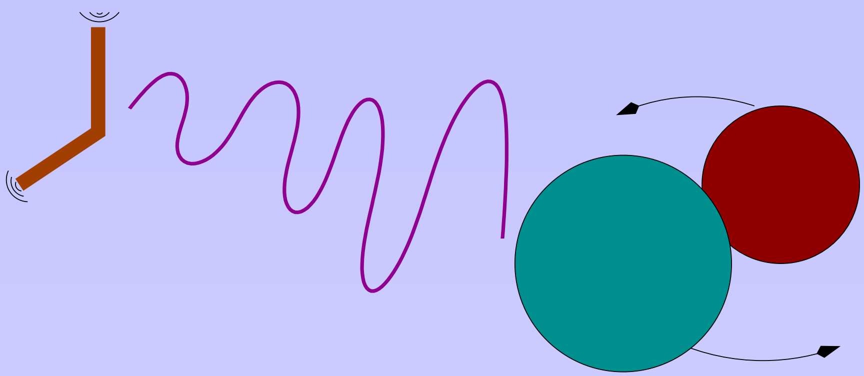
May 2, 2003

Abstract

A formalism for constructing initial data representing black-hole binaries in quasi-equilibrium is developed. If each black hole is assumed to be in quasi-equilibrium, then conditions on the shear and expansion of the outgoing null rays at each black hole yield a set of boundary conditions for the initial data variables. This formalism should allow for the construction of completely general quasi-equilibrium black hole binary initial data. Initial numerical results using this approach will be examined and compared with previous numerical and post-Newtonian results.

Collaborators: Harald Pfeiffer & Saul Teukolsky (Cornell)

Motivation



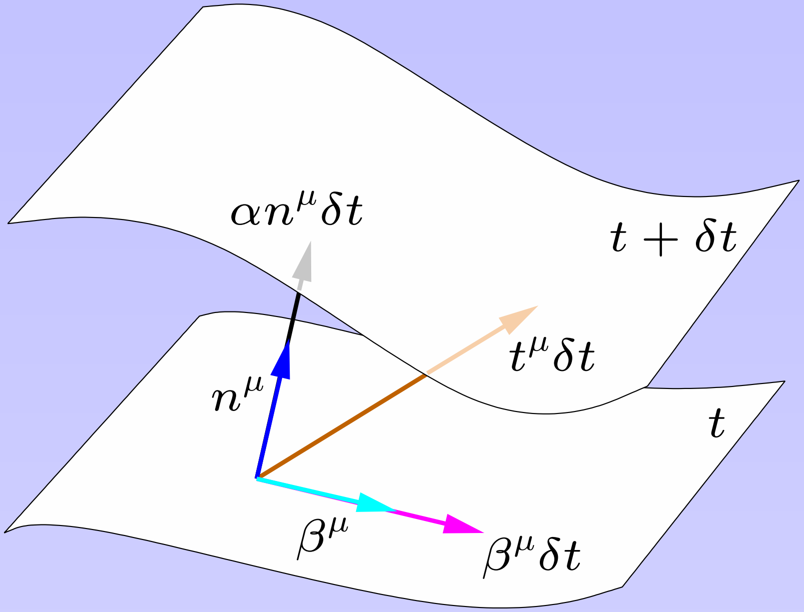
- Black hole binaries are among the most likely sources for early detection with LIGO, VIRGO, GEO,. . .
- Available computed waveforms should increase chance of detecting collision events.

Quasi-Equilibrium Binary Data

- General Relativity doesn't permit *true* equilibrium for astrophysical binary systems.
- When the bodies are sufficiently far apart, the timescale for orbital decay is much larger than the orbital period.
- If the orbit is nearly circular (*quasi-circular*) then there is a *corotating reference frame* in which the binary appears to be at rest.

⋆ Quasi-equilibrium gives us a *physical condition* to guide us in fixing boundary conditions and data that is not otherwise constrained.

The $3+1$ Decomposition



Lapse : α Spatial metric : γ_{ij}

Shift vector : β^i Extrinsic Curvature : K_{ij}

Time vector : $t^\mu = \alpha n^\mu + \beta^\mu$

$$\mathrm{d}s^2 = -\alpha^2 \mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + \beta^i \mathrm{d}t)(\mathrm{d}x^j + \beta^j \mathrm{d}t)$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \qquad K_{\mu\nu} = -\tfrac{1}{2}\gamma^\alpha_\mu \gamma^\beta_\nu \mathcal{L}_n g_{\alpha\beta}$$

Constraint equations

$$\bar{R} + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

$$\bar{\nabla}_j\Big(K^{ij} - \gamma^{ij}K\Big) = 8\pi j^i$$

$$S_{\mu\nu} \equiv \gamma^\alpha_\mu \gamma^\beta_\nu T_{\alpha\beta}$$

$$j_\mu \equiv -\gamma^\nu_\mu n^\alpha T_{\nu\alpha}$$

$$\rho \equiv n^\mu n^\nu T_{\mu\nu}$$

$$T_{\mu\nu} = S_{\mu\nu} + 2n_{(\mu}j_{\nu)} + n_\mu n_\nu \rho$$

Evolution equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i$$

$$\partial_t K_{ij} = -\bar{\nabla}_i \bar{\nabla}_j \alpha + \alpha\Big[\bar{R}_{ij} - 2K_{i\ell}K^\ell_j + K K_{ij} - 8\pi S_{ij} + 4\pi\gamma_{ij}(S - \rho)\Big] + \beta^\ell \bar{\nabla}_\ell K_{ij} + K_{i\ell}\bar{\nabla}_j \beta^\ell + K_{j\ell}\bar{\nabla}_i \beta^\ell$$

Generalized Conformal/TT Initial-Data Decomposition[13]

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad : \quad K^{ij} = \psi^{-10}\tilde{A}^{ij} + \tfrac{1}{3}\psi^{-4}\tilde{\gamma}^{ij}K \quad : \quad \tilde{A}^{ij} \equiv \frac{1}{\tilde{\sigma}}(\tilde{\mathbb{L}}V)^{ij} + \tilde{M}^{ij}$$

$$\tilde{\sigma} \equiv \psi^{-6}\sigma \quad : \quad \tilde{\nabla}^2\psi - \tfrac{1}{8}\psi\tilde{R} - \tfrac{1}{12}\psi^5 K^2 + \tfrac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = -2\pi\psi^5\rho$$

$$\tilde{\nabla}_j(\tilde{\mathbb{L}}V)^{ij} - (\tilde{\mathbb{L}}V)^{ij}\tilde{\nabla}_j \ln\tilde{\sigma} - \tfrac{2}{3}\tilde{\sigma}\psi^6\tilde{\nabla}^i K = -\tilde{\sigma}\tilde{\nabla}_j\tilde{M}^{ij} + 8\pi\tilde{\sigma}\psi^{10}j^i$$

$\tilde{M}^{ij}$ is symmetric-tracefree, but not divergenceless. The variable V^i incorporates the solution of the constraints and the decomposition of $\tilde{M}^{ij}$ into $\tilde{M}^{ij}_{TT}$.

$\tilde{\sigma} = 1 \Rightarrow$ *Conf-TT Method (Method A)*
$\sigma = 1 \Rightarrow$ *Phys-TT Method (Method B)*

$\tilde{M}^{ij} \Rightarrow -\frac{1}{2\tilde{\alpha}}\tilde{u}^{ij}$ *Conf. Thin Sandwich*
$\tilde{\sigma} \Rightarrow 2\tilde{\alpha}$

Conformal Thin Sandwich(TS)[16] $\qquad [\tilde{u}^{ij} \equiv -\partial_t\tilde{\gamma}^{ij}]$

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad : \quad K^{ij} = \psi^{-10}\tilde{A}^{ij} + \tfrac{1}{3}\psi^{-4}\tilde{\gamma}^{ij}K \quad : \quad \tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}}\left((\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij}\right)$$

$$\tilde{\alpha} \equiv \psi^{-6}\alpha \quad : \quad \tilde{\nabla}^2\psi - \tfrac{1}{8}\psi\tilde{R} - \tfrac{1}{12}\psi^5 K^2 + \tfrac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = -2\pi\psi^5\rho$$

$$\tilde{\nabla}_j(\tilde{\mathbb{L}}\beta)^{ij} - (\tilde{\mathbb{L}}\beta)^{ij}\tilde{\nabla}_j \ln\tilde{\alpha} - \tfrac{4}{3}\tilde{\alpha}\psi^6\tilde{\nabla}^i K = \tilde{\alpha}\tilde{\nabla}_j\left(\frac{1}{\tilde{\alpha}}\tilde{u}^{ij}\right) + 16\pi\tilde{\alpha}\psi^{10}j^i$$

$$\tilde{\nabla}^2(\psi^7\tilde{\alpha}) - \tfrac{1}{8}\psi^7\tilde{\alpha}\tilde{R} - \tfrac{5}{12}\psi^{11}\tilde{\alpha}K^2 - \tfrac{7}{8}\psi^{-1}\tilde{\alpha}\tilde{A}_{ij}\tilde{A}^{ij} - \psi^5\beta^i\tilde{\nabla}_i K = -2\pi\psi^{11}\tilde{\alpha}K(\rho + 2S) - \psi^5\partial_t K$$

α and β^i are the lapse and shift. $\tilde{u}^{ij}$ is symmetric-tracefree.

Degrees of Freedom

Kinematical variables

- Lapse α : 1 degree of freedom
- Shift β^i : 3 degrees of freedom

Initial-data variables

- Metric γ_{ij} : 6 degrees of freedom
- Extrinsic curvature K_{ij} : 6 degrees of freedom

Decomposition of initial-data variables

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \begin{cases} \psi & : \quad \text{1 constrained DOF} \\ \tilde{\gamma}_{ij} & : \left\langle \begin{array}{l} \text{3 spatial gauge DOF} \\ \text{2 dynamical DOF} \end{array} \right\} \textit{Freely Specifiable} \end{cases}$$

$$K^{ij} = \psi^{-10}\left[\frac{1}{\tilde{\sigma}}(\tilde{\mathbb{L}}V)^{ij} + \tilde{M}^{ij}\right] + \tfrac{1}{3}\psi^{-4}\tilde{\gamma}^{ij}K \begin{cases} V^i & : \left\langle \begin{array}{l} (\tilde{\mathbb{L}}V)^{ij} \equiv \tilde{\nabla}^i V^j + \tilde{\nabla}^i V^j - \frac{1}{3}\tilde{\gamma}^{ij}\tilde{\nabla}_k V^k \\ \text{3 constrained DOF} \end{array} \right. \\ \tilde{M}^{ij} & : \left\langle \begin{array}{l} \tilde{\nabla}_j \tilde{M}^{ij} = \tilde{\nabla}_j\left(\frac{1}{\tilde{\sigma}}(\tilde{\mathbb{L}}X)^{ij}\right) \\ \text{2 dynamical DOF} \end{array} \right. \\ K & : \text{1 temporal gauge DOF} \\ \tilde{\sigma} & : \textit{Defn. of TT decomp.} \end{cases} \quad \textit{Freely Specifiable}$$

"Traditional" Black-Hole Data

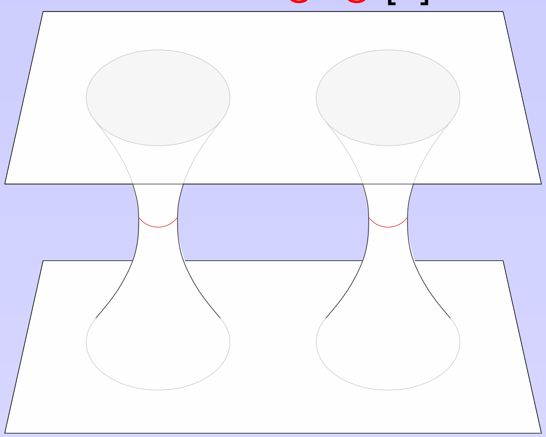
Conformal flatness and maximal slicing

$$\left.\begin{aligned} \tilde{\gamma}_{ij} &= f_{ij}\ \textit{(flat)} \\ \tilde{M}^{ij} &= 0 \\ K &= 0 \\ \tilde{\sigma} &= 1 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &\tilde{\nabla}_i(\tilde{\mathbb{L}}V)^{ij} = 0 \quad \Rightarrow \\ &\tilde{\nabla}^2\psi + \tfrac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = 0 \end{aligned}\right.$$

Bowen-York solution[4]
Analytic solutions for $\tilde{A}^{ij}$

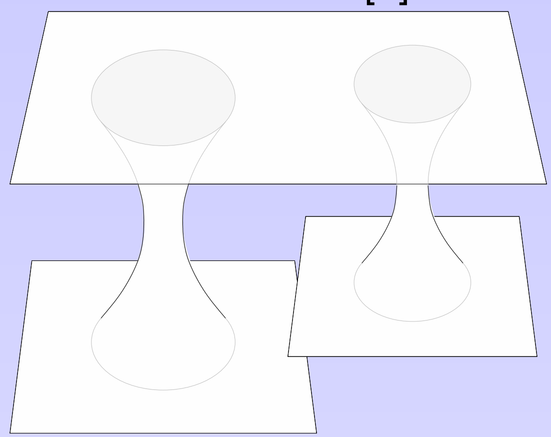
Three general solution schemes

Conformal Imaging-[7]



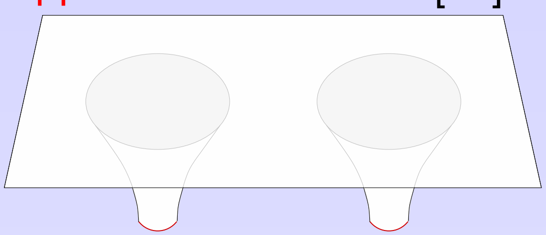
Inversion symmetry inner-BC

Puncture Method-[5]



No inner-BC: singular behavior factored out

Apparent Horizon BC-[14]



Apparent horizon inner-BC

All methods can produce very general configurations of multiple black holes, but are fundamentally limited by choices for $\tilde{\gamma}_{ij}$ and Bowen-York $\tilde{A}^{ij}$.

"Better" Black-Hole Data

What is wrong with "traditional" BH initial data?

- Results disagree with PN predictions for black holes in quasi-circular orbits.
- There is no control of the initial "wave" content.
- Spinning holes are not represented well.

How do we construct improved BH initial data?

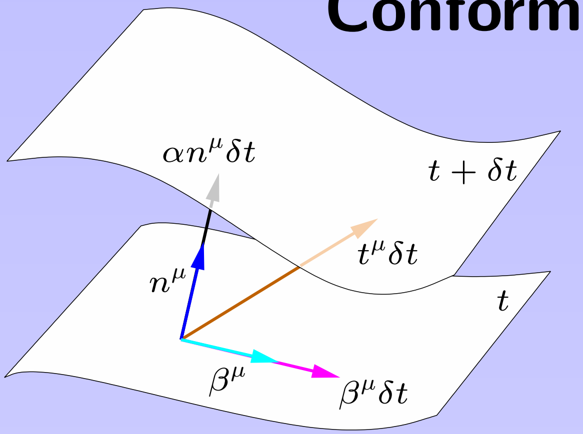
We must carefully choose the

- initial dynamical degrees of freedom [in $\tilde{\gamma}_{ij}$ and $\tilde{M}^{ij}$]
- initial temporal and spatial gauge degrees of freedom [in $\tilde{\gamma}_{ij}$ and K]
- boundary conditions on the constrained degrees of freedom [in ψ and V^i]
- TT decomposition weight function [$\tilde{\sigma}$]

so as to conform to the desired physical content of the initial data.

- For black holes in quasi-circular orbits, we can use the principle of *quasi-equilibrium* to guide our choices.
- Quasi-equilibrium is a *dynamical* concept and we can simplify our task by choosing a decomposition of the initial-data variables that has connections to dynamics.

Conformal Thin-Sandwich Decomposition



$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$$

$$K^{ij} = \frac{\psi^{-10}}{2\tilde{\alpha}}\left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij}\right] + \tfrac{1}{3}\psi^{-4}\tilde{\gamma}^{ij}K$$

Hamiltonian Const. $\tilde{\nabla}^2\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{12}\psi^5 K^2 + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = -2\pi\psi^5\rho$

Momentum Const. $\tilde{\nabla}_j(\tilde{\mathbb{L}}\beta)^{ij} - (\tilde{\mathbb{L}}\beta)^{ij}\tilde{\nabla}_j\tilde{\alpha} = \frac{4}{3}\tilde{\alpha}\psi^6\tilde{\nabla}^i K + \tilde{\alpha}\tilde{\nabla}_j\left(\frac{1}{\tilde{\alpha}}\tilde{u}^{ij}\right) + 16\pi\tilde{\alpha}\psi^{10}j^i$

Const. Tr(K) eqn. $\tilde{\nabla}^2(\psi^7\tilde{\alpha}) - (\psi^7\tilde{\alpha})\left[\frac{1}{8}\psi\tilde{R} + \frac{5}{12}\psi^5K^2 + \frac{7}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} - \psi^5\beta^i\tilde{\nabla}_iK\right]$

$$= -2\pi\psi^5K(\rho + 2S) - \psi^5\partial_t K$$

$$\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}}\left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij}\right]$$

Constrained vars : ψ and β^i

Freely specified : $\tilde{\gamma}_{ij}$, $\tilde{u}^{ij}$, K, and $\partial_t K$

$\tilde{u}^{ij}$ and β^i have a simple physical interpretation, unlike $\tilde{M}^{ij}$ and V^i.

Quasi-equilibrium $\Rightarrow \begin{cases} \tilde{u}^{ij} = 0 \\ \partial_t K = 0 \end{cases}$

Equations of Quasi-Equilibrium

Ham. & Mom. const. eqns., & Const Tr(K) eqn. from Conf. TS $+ \tilde{u}^{ij} = \partial_t K = 0$ $\Bigg\}$ $\Rightarrow$ **Eqns. of Quasi-Equilibrium**

With $\tilde{\gamma}_{ij} = f_{ij}$ and $K = 0$, these equations have been widely used to construct binary neutron star initial data[2, 12, 3, 15].

Binary neutron star initial data require:

- boundary conditions at infinity compatible with asymptotic flatness and corotation.

$$\psi|_{\mathrm{r}\to\infty} = 1 \quad \beta^i|_{\mathrm{r}\to\infty} = \Omega\left(\frac{\partial}{\partial\phi}\right)^i \quad \alpha|_{\mathrm{r}\to\infty} = 1$$

- compatible solution of the equations of hydrostatic equilibrium. $(\Rightarrow \Omega)$

Binary black hole initial data require:

- a means for choosing the angular velocity of the orbit Ω.

⋆ *with excision*, inner boundary conditions are needed for ψ, β^i, and $\tilde{\alpha}$.

Gourgoulhon, Grandclément, & Bonazzola[10, 11]: Black-hole binaries with $\tilde{\gamma}_{ij} = f_{ij}$ & $K = 0$, "inversion-symmetry", and "Killing-horizon" conditions on the excision boundaries.

"Solutions" require *constraint violating* regularity condition imposed on inner boundaries!

Constructing Regular Binary Black Hole QE ID

Why does the GGB approach have problems?

- Inversion-symmetry demands $\tilde{\alpha} = 0$ on the inner boundaries.

- Inversion-symmetry BC's are not compatible with $(\tilde{\mathbb{L}}\beta)^{ij} = 0$ **and** $\tilde{u}^{ij} = 0$, unless ∂_t is a true Killing vector. [conjecture]

$$\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}}\left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij}\right]$$

How do we proceed?

⋆ Eliminate dependence on inversion symmetry by letting the physical condition of quasi-equilibrium dictate the boundary conditions.

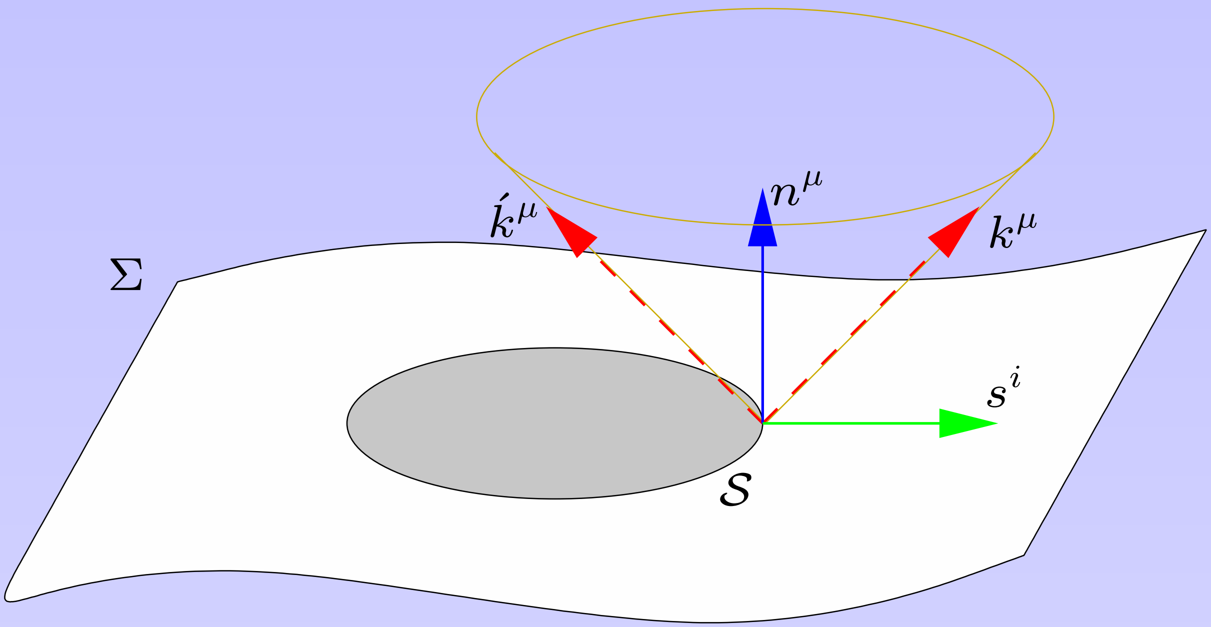
- Find a method that allows for general choices of $\tilde{\gamma}_{ij}$ & K.

Approach

- Demand that the excision *(inner)* boundary be an *apparent horizon*.

- Demand that the apparent horizon be in quasi-equilibrium.

The Inner Boundary



$$s_i \equiv \frac{\bar{\nabla}_i \tau}{|\bar{\nabla}\tau|}$$

$$h_{ij} \equiv \gamma_{ij} - s_i s_j$$

$$k^\mu \equiv \frac{1}{\sqrt{2}}(n^\mu + s^\mu)$$

$$\acute{k}^\mu \equiv \frac{1}{\sqrt{2}}(n^\mu - s^\mu)$$

Extrinsic curvature of $\mathcal{S}$ embedded in spacetime

$$\Sigma_{\mu\nu} \equiv \tfrac{1}{2} h^\alpha_\mu h^\beta_\nu \mathcal{L}_k g_{\alpha\beta}$$

$$\acute{\Sigma}_{\mu\nu} \equiv \tfrac{1}{2} h^\alpha_\mu h^\beta_\nu \mathcal{L}_{\acute{k}} g_{\alpha\beta}$$

Extrinsic curvature of $\mathcal{S}$ embedded in Σ

$$H_{ij} \equiv \tfrac{1}{2} h^k_i h^\ell_j \mathcal{L}_s \gamma_{k\ell}$$

$$\Sigma_{ij} = \frac{1}{\sqrt{2}}(H_{ij} - J_{ij})$$

$$\acute{\Sigma}_{ij} = -\frac{1}{\sqrt{2}}(H_{ij} + J_{ij})$$

Projections of K_{ij} onto $\mathcal{S}$

$$J_{ij} \equiv h^k_i h^\ell_j K_{k\ell}$$

$$J_i \equiv h^k_i s^\ell K_{k\ell}$$

$$J \equiv h^{ij} J_{ij} = h^{ij} K_{ij}$$

Expansion of null rays

$$\theta \equiv h^{ij}\Sigma_{ij} = \frac{1}{\sqrt{2}}(H - J)$$

$$\acute{\theta} \equiv h^{ij}\acute{\Sigma}_{ij} = -\frac{1}{\sqrt{2}}(H + J)$$

Shear of null rays

$$\sigma_{ij} \equiv \Sigma_{ij} - \tfrac{1}{2} h_{ij}\theta$$

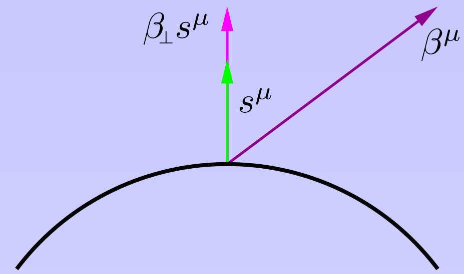
$$\acute{\sigma}_{ij} \equiv \acute{\Sigma}_{ij} - \tfrac{1}{2} h_{ij}\acute{\theta}$$

AH and QE Conditions on the Inner Boundary

The quasi-equilibrium inner boundary conditions start with the following assumptions:

1. The inner boundary $\mathcal{S}$ is a (MOTS):
   marginally outer-trapped surface

   $\longrightarrow \quad \theta = 0$

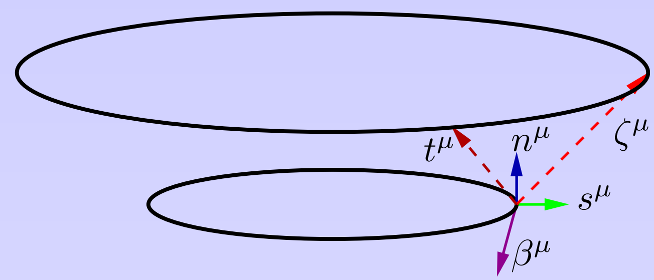


2. The inner boundary $\mathcal{S}$ doesn't move:

   $\longrightarrow \quad \mathcal{L}_\zeta \tau = 0$ and $D_i \mathcal{L}_\zeta \tau \equiv h_i^j \bar{\nabla}_j \mathcal{L}_\zeta \tau = 0$

$$t^\mu = \alpha n^\mu + \beta^\mu \qquad \zeta^\mu \equiv \alpha n^\mu + \beta_\perp s^\mu$$

$$\beta_\perp \equiv \beta^i s_i$$



3. The inner boundary $\mathcal{S}$ remains a MOTS:

   $\longrightarrow \quad \mathcal{L}_\zeta \theta = 0$

4. The horizons are in quasi-equilibrium:

   $\longrightarrow \quad \sigma_{ij} = 0$ and no matter is on $\mathcal{S}$

AH/Quasi-Equilibrium Boundary Conditions

$$\theta = \frac{\psi^{-2}}{\sqrt{2}}\left[\tilde{h}^{ij}\tilde{\nabla}_i\tilde{s}_j + 4\tilde{s}^k\tilde{\nabla}_k \ln\psi - \psi^2 J\right]$$

$$\begin{aligned}\mathcal{L}_\zeta\theta &= -\tfrac{1}{\sqrt{2}}\left[\theta(\theta + \tfrac{1}{2}\acute{\theta} + \tfrac{1}{\sqrt{2}}K) + \mathcal{E}\right](\beta_\perp + \alpha) \\ &\quad - \tfrac{1}{\sqrt{2}}\left[\theta(\tfrac{1}{2}\theta - \tfrac{1}{2}\acute{\theta} + \tfrac{1}{\sqrt{2}}K) + \mathcal{D} + 8\pi T_{\mu\nu}k^\mu \acute{k}^\nu\right](\beta_\perp - \alpha) \\ &\quad + \theta s^i \bar{\nabla}_i \alpha\end{aligned}$$

$$\begin{aligned}\mathcal{D} &\equiv h^{ij}(D_i + J_i)(D_j + J_j) - \tfrac{1}{2}{}^2R \\ \mathcal{E} &\equiv \sigma_{ij}\sigma^{ij} + 8\pi T_{\mu\nu}k^\mu k^\nu\end{aligned}$$

$$\begin{aligned}\sigma_{ij} &= \frac{1}{\sqrt{2}}(H_{ij} - \tfrac{1}{2}h_{ij}H)\left(1 - \frac{\beta_\perp}{\alpha}\right) \\ &\quad - \frac{1}{\sqrt{2}}\frac{\psi^4}{\alpha}\left\{\tilde{D}_{(i}\beta_{\parallel j)} - \tfrac{1}{2}\tilde{h}_{ij}\tilde{D}_k\beta_\parallel{}^k - \tfrac{1}{2}[\tilde{h}_{ik}\tilde{h}_{j\ell}\tilde{u}^{k\ell} - \tfrac{1}{2}\tilde{h}_{ij}\tilde{h}_{k\ell}\tilde{u}^{k\ell}]\right\}\end{aligned}$$

$$\begin{aligned}\tilde{s}^k\tilde{\nabla}_k \ln\psi &= -\tfrac{1}{4}(\tilde{h}^{ij}\tilde{\nabla}_i\tilde{s}_j - \psi^2 J) \\ \beta^i &= \alpha\psi^{-2}\tilde{s}^i + \beta^i_\parallel \\ 0 &= \tilde{D}_{(i}\beta_{\parallel j)} - \tfrac{1}{2}\tilde{h}_{ij}\tilde{D}_k\beta_\parallel{}^k\end{aligned}$$

Defining the Spin of the Black Hole

The spin parameters ${\beta_{\|}}^i$ can be defined by demanding that the time vector associated with quasi-equilibrium in the corotating frame must be null, forming the null generators of the horizon.

$$k^\mu \propto (n^\mu + s^\mu) \implies k^\mu = \left[1, \alpha s^i - \beta^i\right]$$

This vector k^μ is null for any choice of α & β^i.

In the frame where a black hole is not spinning, the null time vector has components $t^\mu = [1, \vec{0}]$.

Corotating Holes

Corotating holes are at rest in the corotating frame, where we must pose boundary conditions. So,

$$k^\mu = \left[1, \alpha s^i - \beta^i\right] = [1, \vec{0}]$$

Thus we find

$$\beta^i = \alpha s^i \;\Rightarrow\; {\beta_{\|}}^i = 0$$

Irrotational Holes

Irrotational holes are at rest in the inertial frame. With the time vectors in the inertial and corotating frames related by

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} + \Omega \frac{\partial}{\partial \phi}$$

$$k^\mu = \left[1, \alpha s^i - \beta^i\right] = [1, -\Omega(\partial/\partial\phi)^i]$$

Thus we find

$$\beta^i = \alpha s^i + \Omega \left(\frac{\partial}{\partial \phi}\right)^i \Rightarrow {\beta_{\|}}^i = \Omega \xi^i$$

$$\xi^i \approx \left(\tfrac{\partial}{\partial\phi}\right)^i \quad \& \quad \tilde{D}_{(i}\xi_{j)} - \tfrac{1}{2}\tilde{h}_{ij}\tilde{D}_k\xi^k = 0$$

The Orbital Angular Velocity

- For a given choice of the Lapse BC, $\tilde{\gamma}_{ij}$ and K, we are still left with a family of solutions parameterized by the orbital angular velocity Ω.
- Except for the case of a single spinning black hole, it is not reasonable to expect more than one value of Ω to correspond to a system in quasi-equilibrium.

GGB[10, 11] have suggested a way to pick the quasi-equilibrium value of Ω:

Ω is chosen as the value for which the ADM energy $E_{\rm ADM}$ equals the Komar mass $M_{\rm K}$.

Komar mass	$M_{\rm K} = \frac{1}{4\pi}\oint_\infty \gamma^{ij}(\bar{\nabla}_i\alpha - \beta^k K_{ik})\mathrm{d}^2S_j$	Acceptable definition of the mass *only for stationary spacetimes*.
ADM energy	$E_{\rm ADM} = \frac{1}{16\pi}\oint_\infty \gamma^{ij}\bar{\nabla}_k(\mathcal{G}^k_i - \delta^k_i\mathcal{G})\mathrm{d}^2S_j$	Acceptable definition of the mass *for arbitrary spacetimes*. $\mathcal{G}_{ij} \equiv \gamma_{ij} - f_{ij}$

The Lapse BC & QE

So far, nothing has fixed a boundary condition on the lapse α. One possibility[9] is to recall that $\theta\acute{\theta}$ is a Lorentz invariant and so to consider $\mathcal{L}_\zeta\acute{\theta} = 0$ as a quasi-equilibrium condition.

$$\mathcal{L}_\zeta\acute{\theta} = 0 \qquad \Rightarrow \qquad J\tilde{s}^i\tilde{\nabla}_i\alpha = -\psi^2(J^2 - JK + \tilde{\mathcal{D}})\alpha$$

$$\tilde{\mathcal{D}} \equiv \psi^{-4}[\tilde{h}^{ij}(\tilde{D}_i - J_i)(\tilde{D}_j - J_j) - \tfrac{1}{2}{}^2\tilde{R} + 2\tilde{D}^2\ln\psi]$$

This conditions is satisfied for stationary solutions, but seems to be degenerate with the other QE boundary conditions. To see this, note that the stationary maximal slicings of Schwarzschild form a 1-parameter family:

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\mathrm{d}R^2}{1 - \frac{2M}{R} + \frac{C^2}{R^4}} + R^2\mathrm{d}^2\Omega \\ \alpha &= \sqrt{1 - \frac{2M}{R} + \frac{C^2}{R^4}} \\ \beta^R &= \frac{C}{R^2}\sqrt{1 - \frac{2M}{R} + \frac{C^2}{R^4}} \end{aligned}$$

$$K^i_j = \frac{C}{R^3}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\alpha|_{\mathcal{S}} = \frac{C}{4M^2}$$

Summary of QE Formalism

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \qquad K^{ij} = \psi^{-10}\tilde{A}^{ij} + \tfrac{1}{3}\gamma^{ij}K \qquad \tilde{A}^{ij} = \frac{\psi^6}{2\alpha}(\tilde{\mathbb{L}}\beta)^{ij}$$

$\partial_t \tilde{\gamma}_{ij} = 0$

$$\begin{aligned}
\tilde{\nabla}^2\psi - \tfrac{1}{8}\psi\tilde{R} - \tfrac{1}{12}\psi^5K^2 + \tfrac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} &= 0 \\
\tilde{\nabla}_j(\tilde{\mathbb{L}}\beta)^{ij} - (\tilde{\mathbb{L}}\beta)^{ij}\tilde{\nabla}_j \ln \alpha\psi^{-6} &= \tfrac{4}{3}\alpha\tilde{\nabla}^iK \\
\tilde{\nabla}^2(\alpha\psi) - (\alpha\psi)\Big[\tfrac{1}{8}\tilde{R} + \tfrac{5}{12}\psi^4K^2 + \tfrac{7}{8}\psi^{-8}A_{ij}A^{ij}\Big] &= \psi^5\beta^i\tilde{\nabla}_iK
\end{aligned}$$

$\partial_t K = 0$

$$\tilde{s}^k\tilde{\nabla}_k \ln\psi|_{\mathcal{S}} = -\tfrac{1}{4}(\tilde{h}^{ij}\tilde{\nabla}_i\tilde{s}_j - \psi^2 J)|_{\mathcal{S}}$$

$\theta = 0$

$$\beta^i|_{\mathcal{S}} = \begin{cases} \alpha\psi^{-2}\tilde{s}^i|_{\mathcal{S}} & \text{corotation} \\ \alpha\psi^{-2}\tilde{s}^i|_{\mathcal{S}} + \Omega\xi^i|_{\mathcal{S}} & \text{irrotation} \end{cases}$$

$\mathcal{L}_\zeta\theta = 0$

$\sigma_{ij} = 0$

$\alpha|_{\mathcal{S}} =$ unspecified by QE

$$\begin{aligned}
\psi|_{r\to\infty} &= 1 \\
\beta^i|_{r\to\infty} &= \Omega\left(\frac{\partial}{\partial\phi}\right)^i \\
\alpha|_{r\to\infty} &= 1
\end{aligned}$$

The only remaining freedom in the system is the choice of the lapse boundary condition, the initial spatial and temporal gauge, and the initial dynamical("wave") content found in $\alpha|_{\mathcal{S}}$, $\tilde{\gamma}_{ij}$ and K.

Results

Corotation

$\tilde{\gamma}_{ij} = f_{ij}$: Maximal Slicing:

- $\frac{\partial(\alpha\psi)}{\partial r} = 0$
- $\alpha\psi = \frac{1}{2}$
- $\frac{\partial(\alpha\psi)}{\partial r} = \frac{\alpha\psi}{2r}$

$\tilde{\gamma}_{ij} = f_{ij}$: Eddington-Finkelstein Slicing:

- $\frac{\partial(\alpha\psi)}{\partial r} = 0$
- $\alpha\psi = \frac{1}{2}$
- $\frac{\partial(\alpha\psi)}{\partial r} = \alpha\psi$

Compared with

- Effective-One-Body PN[8]
- Inversion-Symmetric HKV[11]

Irrotation

$\tilde{\gamma}_{ij} = f_{ij}$: Maximal Slicing:

- $\frac{\partial(\alpha\psi)}{\partial r} = 0$
- $\alpha\psi = \frac{1}{2}$
- $\frac{\partial(\alpha\psi)}{\partial r} = \frac{\alpha\psi}{2r}$

$\tilde{\gamma}_{ij} = f_{ij}$: Eddington-Finkelstein Slicing:

- $\frac{\partial(\alpha\psi)}{\partial r} = 0$
- $\alpha\psi = \frac{1}{2}$
- $\frac{\partial(\alpha\psi)}{\partial r} = \alpha\psi$

Compared with

- Effective-One-Body PN[8]
- Conformal Imaging[6]
- Puncture Method[1]

Constructing Evolutionary Sequences

Given quasi-equilibrium data for binary black holes at various separations, how do we connect them together to form an evolutionary sequence?

⋆ There is the freedom to set the overall mass scale for each solution.

- Let $e(s)$, $j(s)$, and $\omega(s)$ denote the numerical solutions along a sequence parameterized by some measure of the separation s.

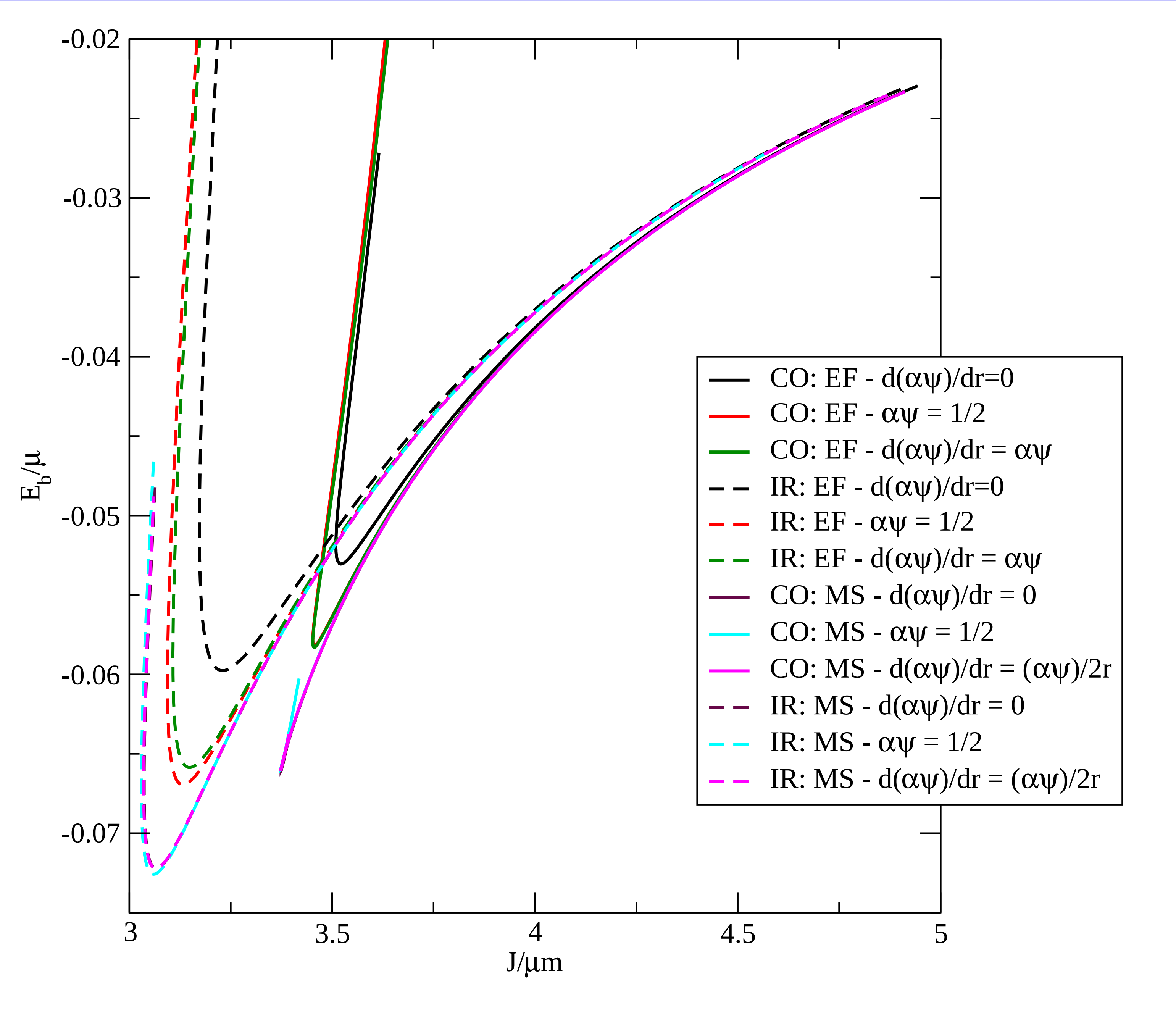
- Let $\chi(s)$ define a mass scaling along the same sequence.

- Define
$$\begin{aligned} E_{\rm ADM}(s) &\equiv \chi(s)m(s) \\ J(s) &\equiv \chi^2(s)j(s) \\ \Omega(s) &\equiv \chi^{-1}(x)\omega(s) \end{aligned}$$

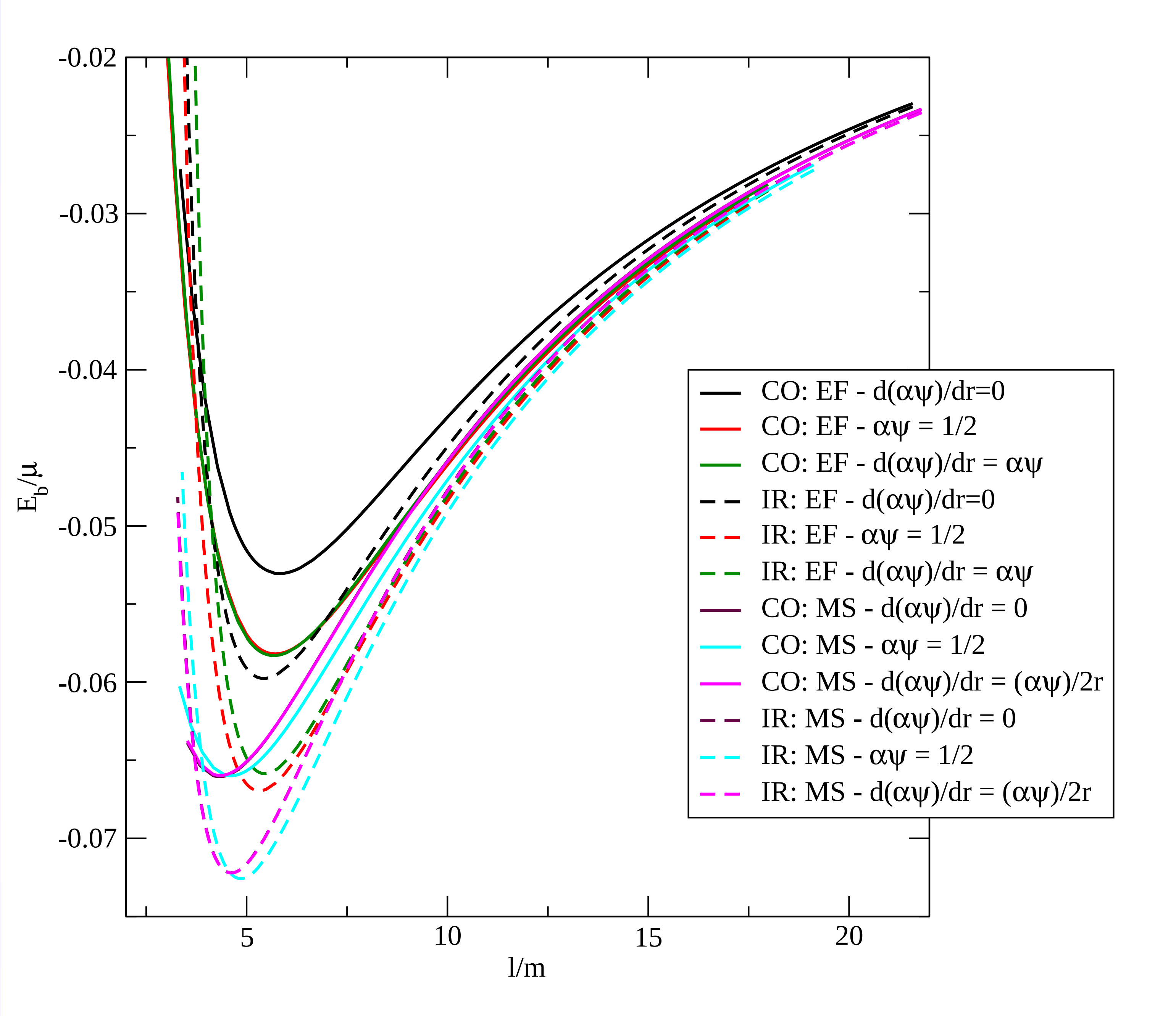
⇒ Choose the mass scale to preserve $\mathrm{d}E_{\rm ADM} = \Omega \mathrm{d}J$.

- Integrating from s_1 to s_2 gives: $\chi(s_2) = \chi(s_1)\exp\left\{-\int_{s_1}^{s_2}\frac{m'-\omega j'}{m-2\omega j}\mathrm{d}s\right\}$

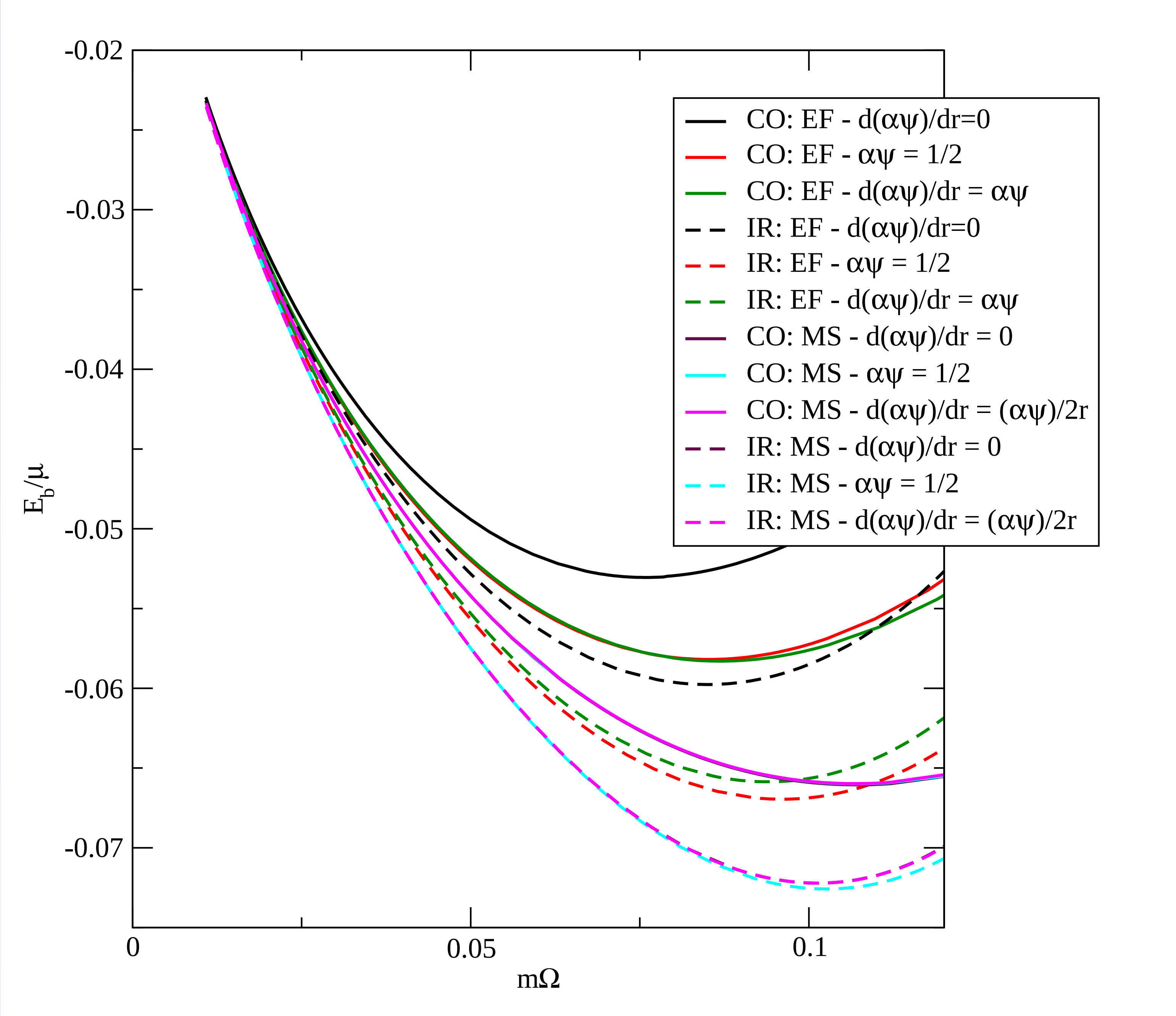
Results: E_b/μ vs $J/\mu m$



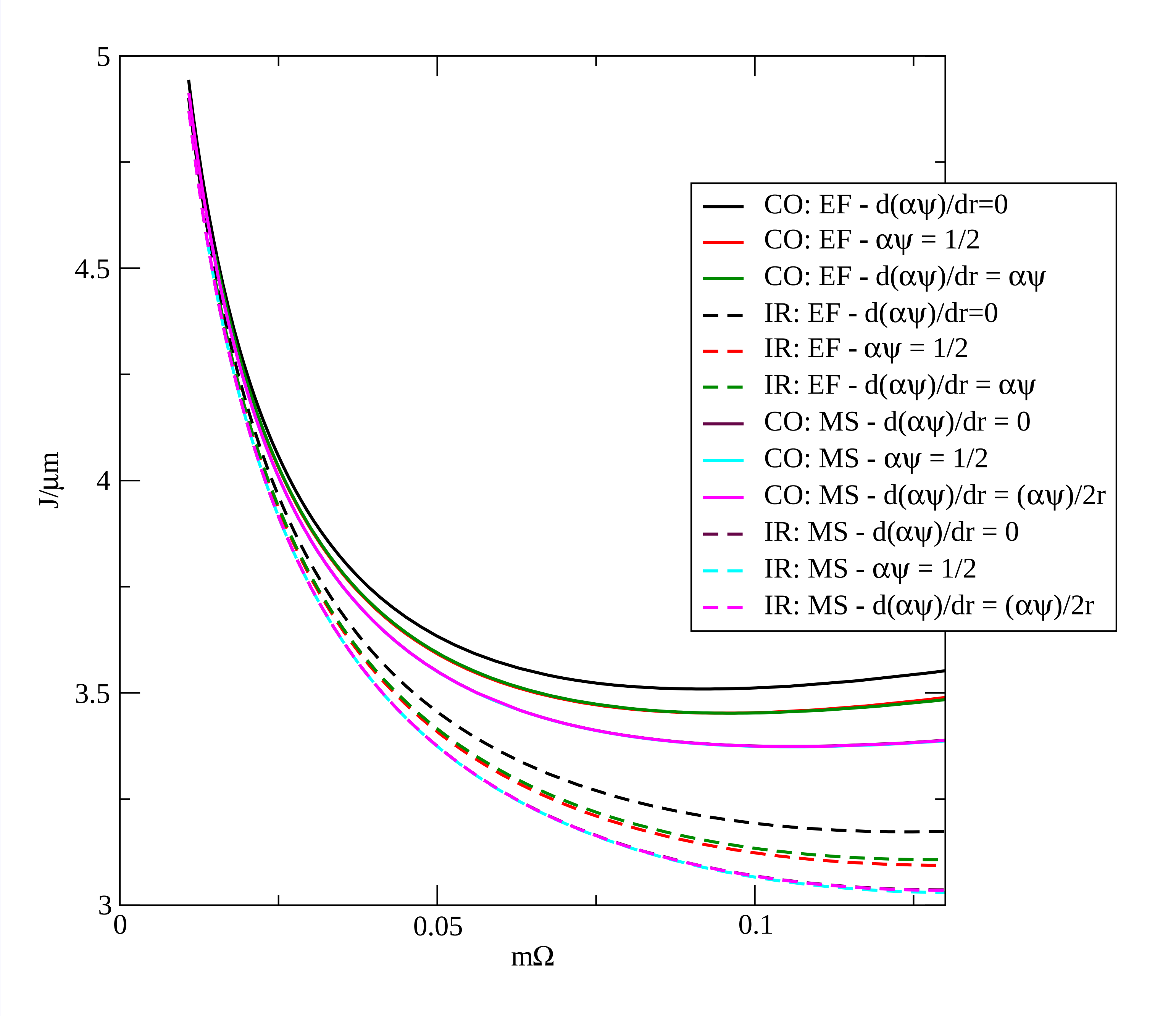
Results: E_b/μ vs ℓ/m



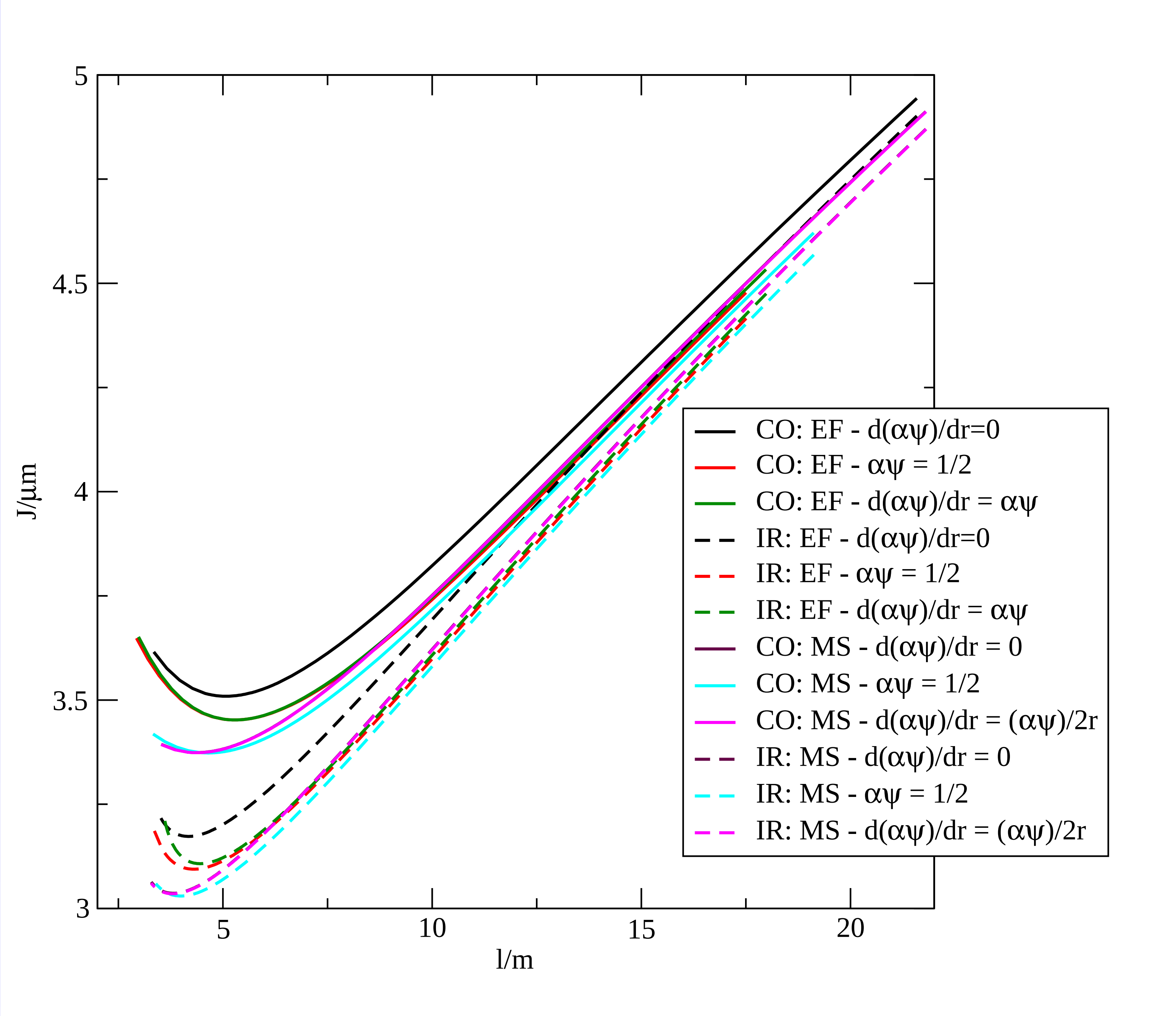
Results: E_b/μ vs $m\Omega$



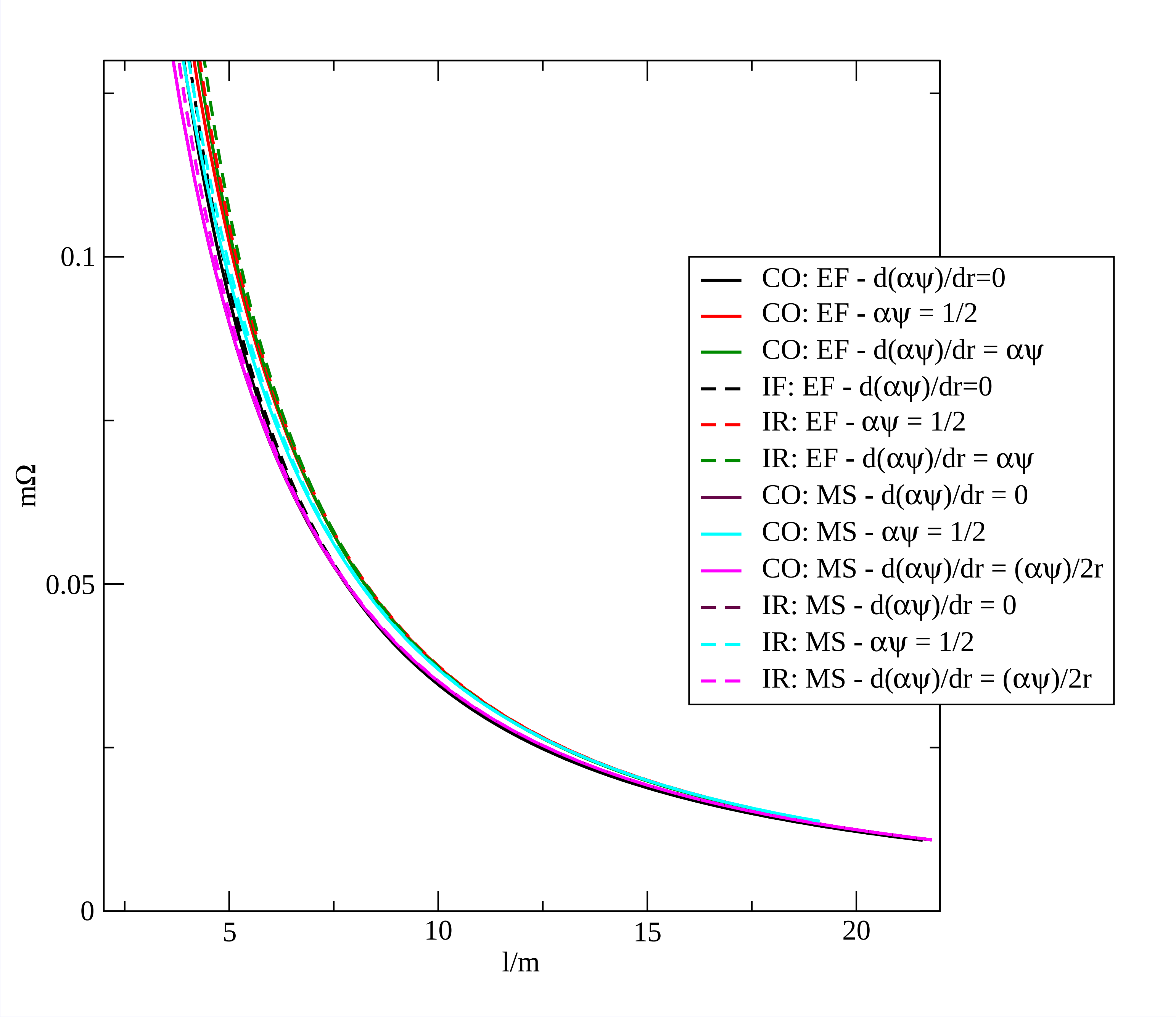
Results: $J/\mu m$ vs $m\Omega$



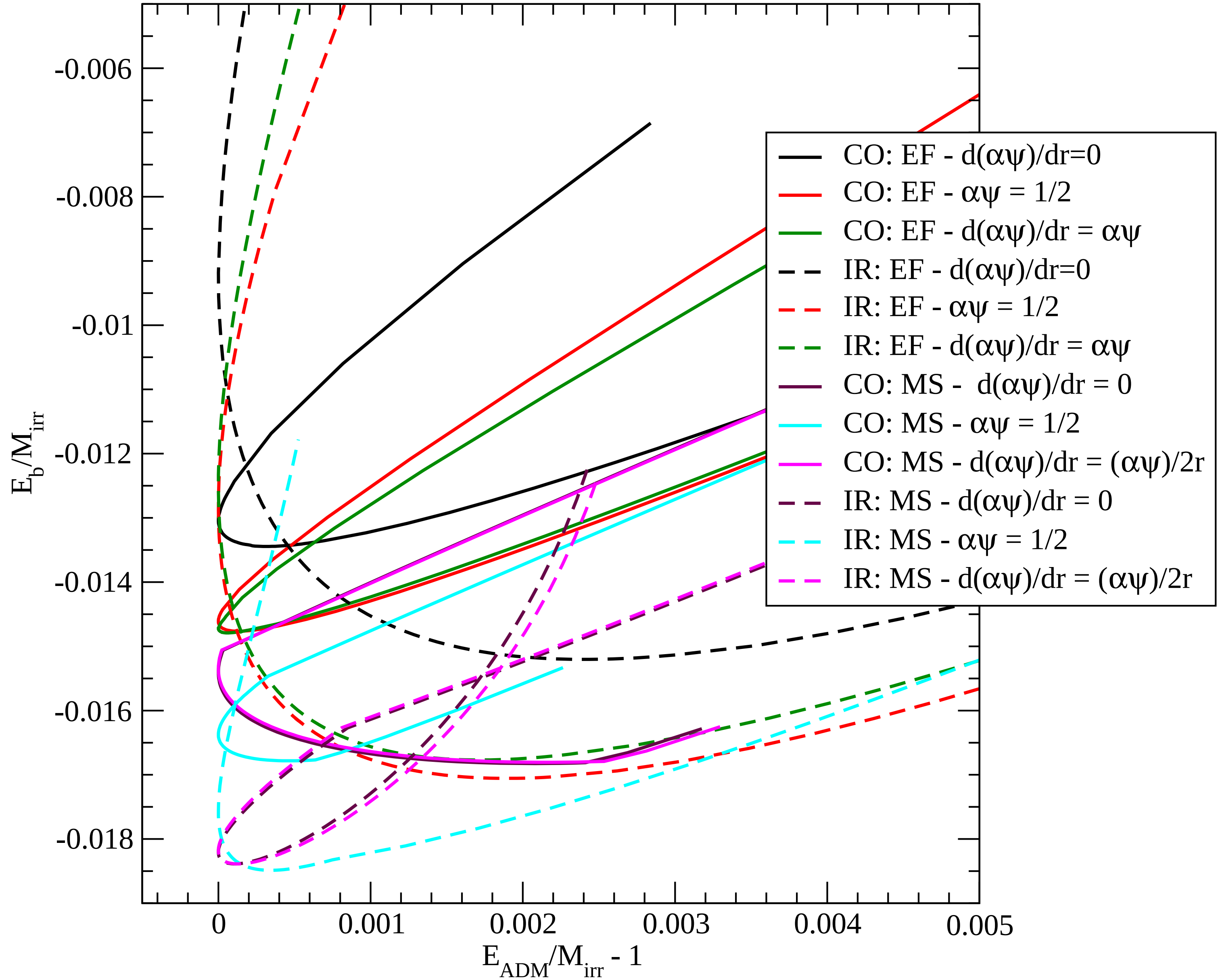
Results: $J/\mu m$ vs ℓ/m



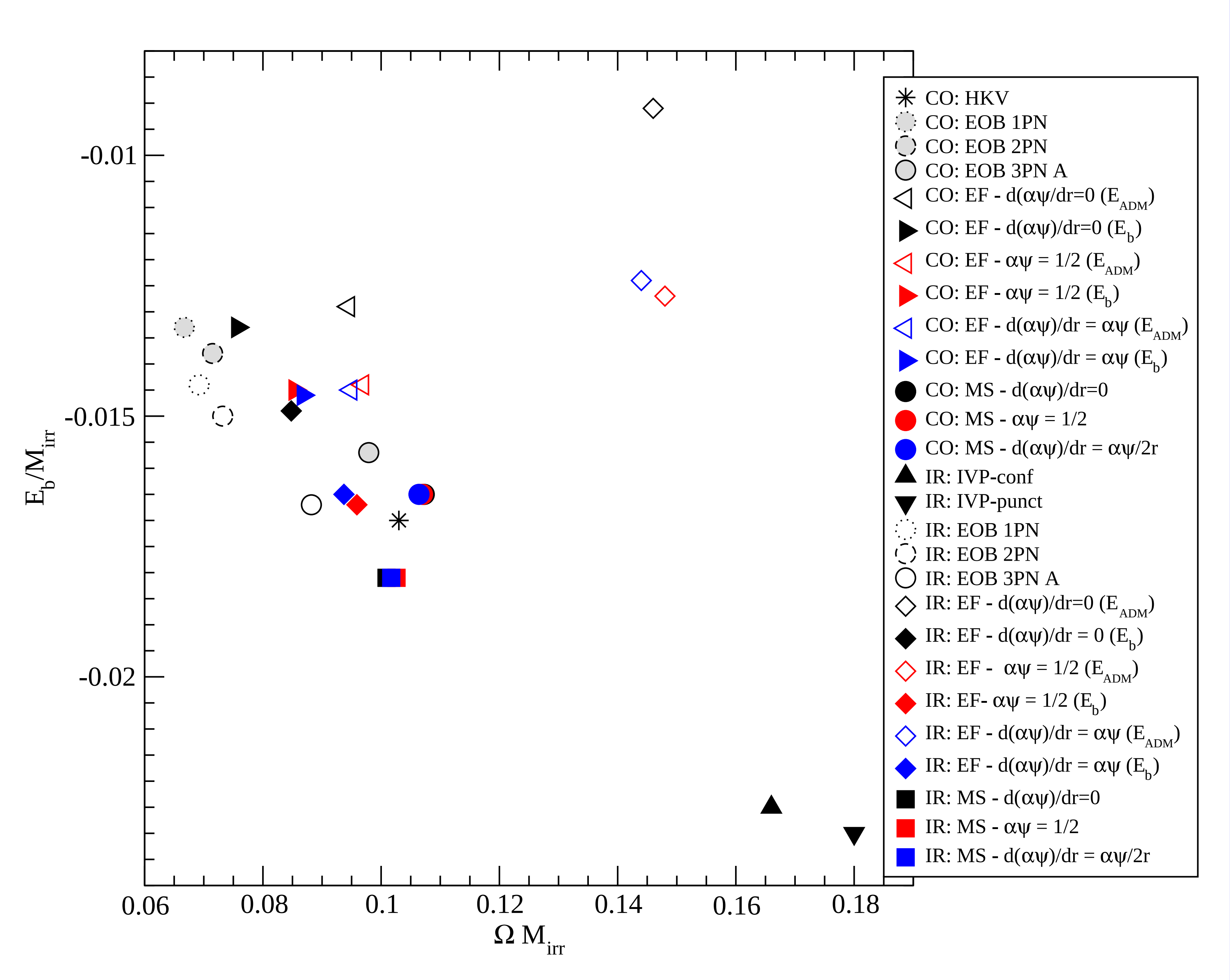
Results: $m\Omega$ vs ℓ/m



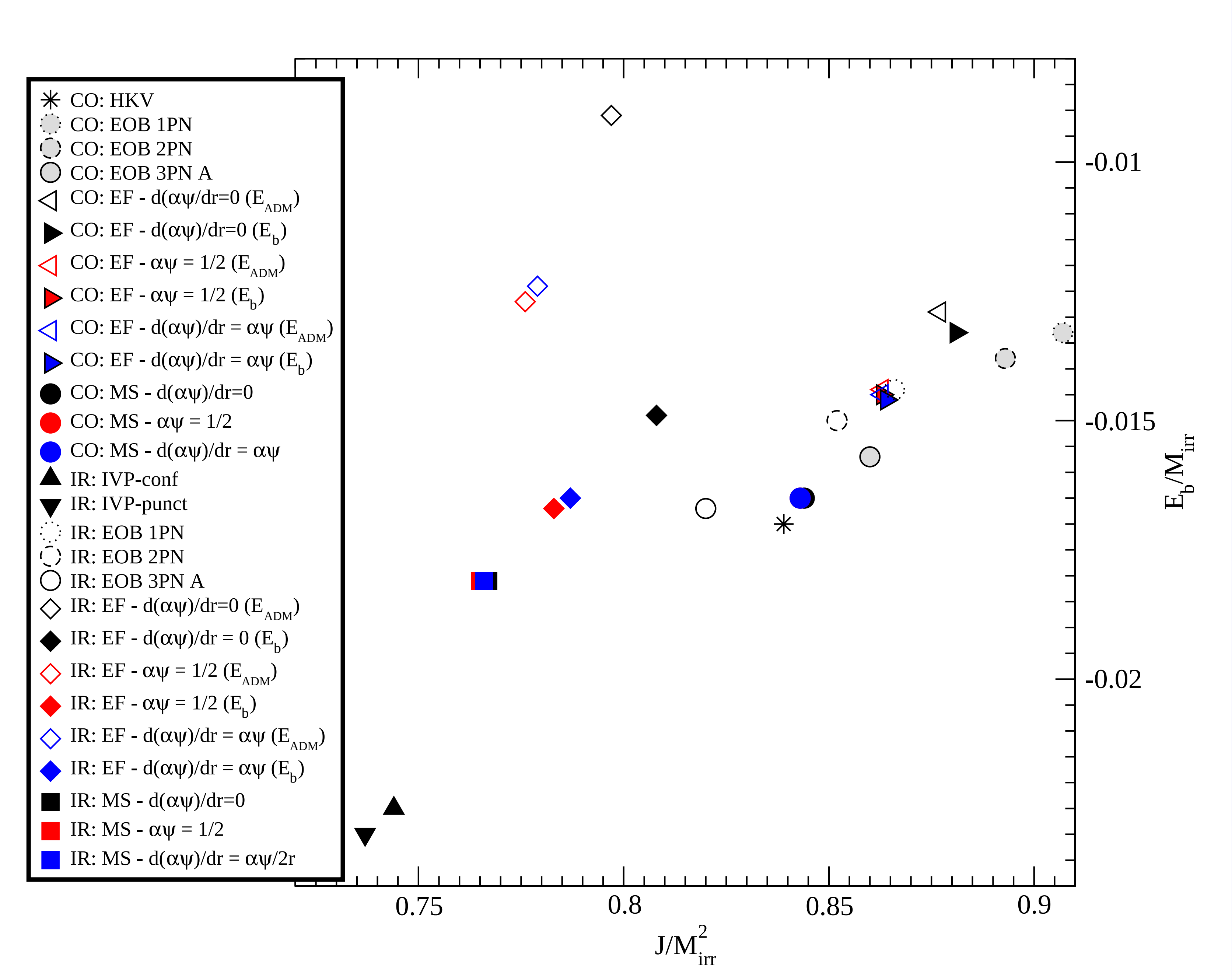
Results: ISCO — E_b/M_{irr} vs $E_{\text{ADM}}/M_{\text{irr}} - 1$



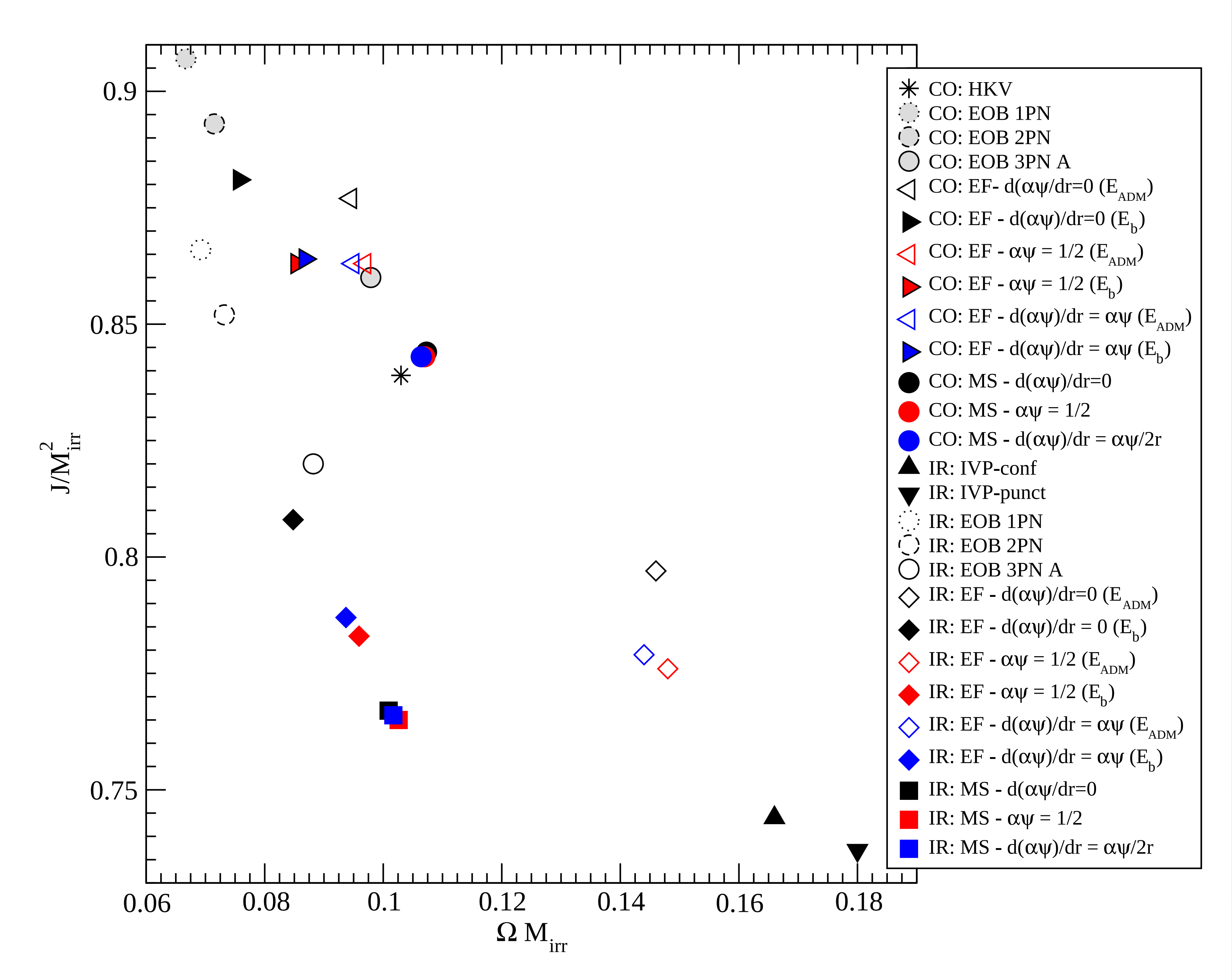
Results: ISCO — E_b/M_{irr} vs ΩM_{irr}



Results: ISCO — E_b/M_{irr} vs J/M_{irr}^2



Results: ISCO — $J/M_{\rm irr}^2$ vs $\Omega M_{\rm irr}$



References

[1] T. W. Baumgarte. Innermost stable circular orbit of binary black holes. *Phys. Rev. D*, 62:024018/1–8, July 2000. 17

[2] T. W. Baumgarte, G. B. Cook, M. A. Scheel, S. L. Shapiro, and S. A. Teukolsky. General relativistic models of binary neutron stars in quasiequilibrium. *Phys. Rev. D*, 57:7299–7311, June 1998. 8

[3] S. Bonazzola, E. Gourgoulhon, and J.-A. Marck. Numerical models of irrotational binary neutron stars in general relativity. *Phys. Rev. Lett.*, 82:892–895, Feb. 1999. 8

[4] J. M. Bowen and J. W. York, Jr. Time-asymmetric initial data for black holes and black-hole collisions. *Phys. Rev. D*, 21:2047–2056, Apr. 1980. 5

[5] S. Brandt and B. Brügmann. A simple construction of initial data for multiple black holes. *Phys. Rev. Lett.*, 78:3606–3609, May 1997. 5

[6] G. B. Cook. Three-dimensional initial data for the collision of two black holes II: Quasicircular orbits for equal mass black holes. *Phys. Rev. D*, 50:5025–5032, Oct. 1994. 17

[7] G. B. Cook, M. W. Choptuik, M. R. Dubal, S. Klasky, R. A. Matzner, and S. R. Oliveira. Three-dimensional initial data for the collision of two black holes. *Phys. Rev. D*, 47:1471–1490, Feb. 1993. 5

[8] T. Damour, E. Gourgoulhon, and P. Grandclément. Circular orbits of corotating binary black holes: Comparison between analytical and numerical results. *Phys. Rev. D*, 66:024007/1–15, July 2002. 17

[9] D. M. Eardley. Black hole boundary conditions and coordinate conditions. *Phys. Rev. D*, 57:2299–2304, Feb. 1998. 15

[10] E. Gourgoulhon, P. Grandclément, and S. Bonazzola. Binary black holes in circular orbits. I. A global spacetime approach. *Phys. Rev. D*, 65:044020/1–19, Feb. 2002. 8, 14

[11] P. Grandclément, E. Gourgoulhon, and S. Bonazzola. Binary black holes in circular orbits. II. Numerical methods and first results. *Phys. Rev. D*, 65:044021/1–18, Feb. 2002. 8, 14, 17

[12] P. Marronetti, G. J. Mathews, and J. R. Wilson. Binary neutron-star systems: From the Newtonian regime to the last stable orbit. *Phys. Rev. D*, 58:107503/1–4, Nov. 1998. 8

[13] H. P. Pfeiffer and J. W. York, Jr. Extrinsic curvature and the Einstin constraints. *Phys. Rev. D*, 67:044022/1–8, Feb. 2003. 3

[14] J. Thornburg. Coordinate and boundary conditions for the general relativistic initial data problem. *Class. Quantum Gravit.*, 4:1119–1131, Sept. 1987. 5

[15] K. Uryū and Y. Eriguchi. New numerical method for constructing quasiequilibrium sequences of irrotational binary neutron stars in general relativity. *Phys. Rev. D*, 61:124023/1–19, June 2000. 8

[16] J. W. York, Jr. Conformal 'thin-sandwich' data for the initial-value problem of general relativity. *Phys. Rev. Lett.*, 82:1350–1353, Feb. 1999. 3