Toward Astrophysical Black-Hole Binaries

Gregory B. Cook

Wake Forest University

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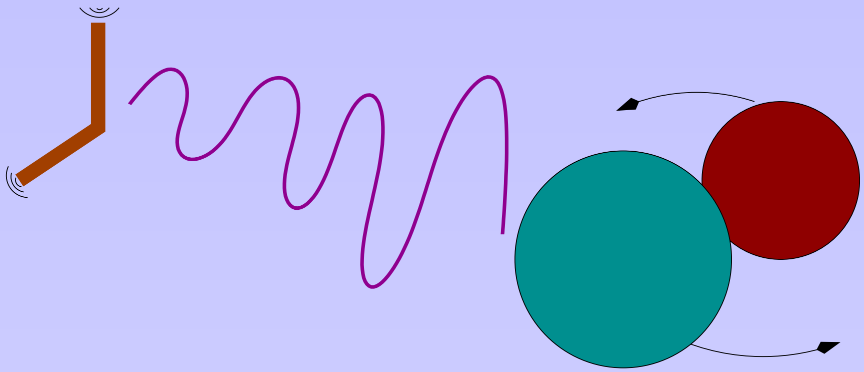
Abstract

A formalism for constructing initial data representing black-hole binaries in quasi-equilibrium is developed. If each black hole is assumed to be in quasi-equilibrium, then a complete set of boundary conditions for *all* initial data variables can be developed. This formalism should allow for the construction of completely general quasi-equilibrium black hole binary initial data.

[6] Related LANL preprint. . .

Collaborators: Harald Pfeiffer & Saul Teukolsky (Cornell)

Motivation



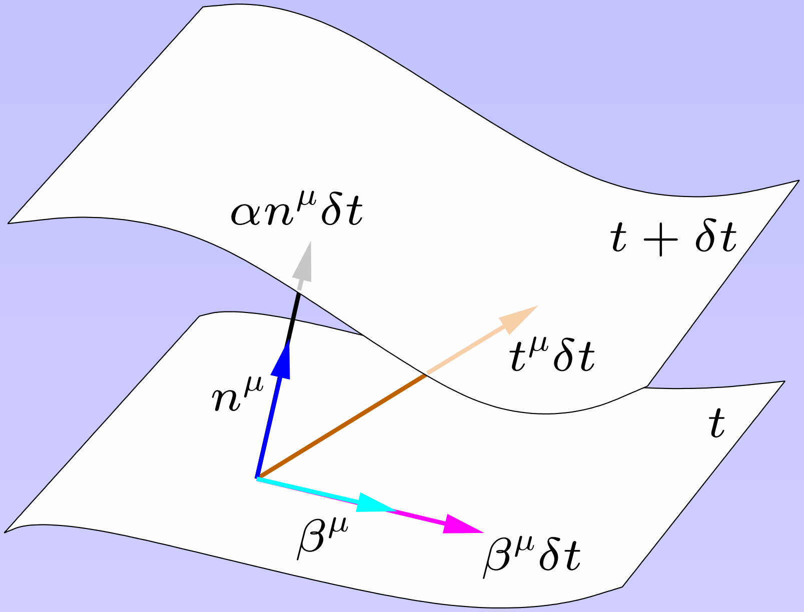
- Black hole binaries are among the most likely sources for early detection with LIGO, VIRGO, GEO,. . .
- Available computed waveforms should increase chance of detecting collision events.

Why Quasi-Equilibrium?

- General Relativity doesn't permit *true* equilibrium for astrophysical binary systems.
- When the bodies are sufficiently far apart, the timescale for orbital decay is much larger than the orbital period.
- If the orbit is nearly circular (*quasi-circular*) then there is a *corotating reference frame* in which the binary appears to be at rest.

⋆ Quasi-equilibrium gives us a *physical condition* to guide us in fixing boundary conditions and data that is not otherwise constrained.

The $3+1$ Decomposition



Lapse : α Spatial metric : γ_{ij}

Shift vector : β^i Extrinsic Curvature : K_{ij}

Time vector : $t^\mu = \alpha n^\mu + \beta^\mu$

$$\mathrm{d}s^2 = -\alpha^2 \mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + \beta^i \mathrm{d}t)(\mathrm{d}x^j + \beta^j \mathrm{d}t)$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \qquad K_{\mu\nu} = -\tfrac{1}{2}\gamma^\alpha_\mu \gamma^\beta_\nu \mathcal{L}_n g_{\alpha\beta}$$

Constraint equations

$$\bar{R} + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

$$\bar{\nabla}_j\left(K^{ij} - \gamma^{ij}K\right) = 8\pi j^i$$

$$S_{\mu\nu} \equiv \gamma^\alpha_\mu \gamma^\beta_\nu T_{\alpha\beta}$$

$$j_\mu \equiv -\gamma^\nu_\mu n^\alpha T_{\nu\alpha}$$

$$\rho \equiv n^\mu n^\nu T_{\mu\nu}$$

$$T_{\mu\nu} = S_{\mu\nu} + 2n_{(\mu}j_{\nu)} + n_\mu n_\nu \rho$$

Evolution equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i$$

$$\partial_t K_{ij} = -\bar{\nabla}_i \bar{\nabla}_j \alpha + \alpha\Big[\bar{R}_{ij} - 2K_{i\ell}K^\ell_j + KK_{ij} - 8\pi S_{ij} + 4\pi\gamma_{ij}(S-\rho)\Big] + \beta^\ell \bar{\nabla}_\ell K_{ij} + K_{i\ell}\bar{\nabla}_j \beta^\ell + K_{j\ell}\bar{\nabla}_i \beta^\ell$$

Degrees of Freedom

Kinematical variables

- Lapse α : 1 degree of freedom
- Shift β^i : 3 degrees of freedom

Initial-data variables

- Metric γ_{ij} : 6 degrees of freedom
- Extrinsic curvature K_{ij} : 6 degrees of freedom

Decomposition of initial-data variables[5] *(Conformal TT decomp)*

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \begin{cases} \psi & : \quad \text{1 constrained DOF} \\ \tilde{\gamma}_{ij} & : \left\langle \begin{array}{l} \text{3 spatial gauge DOF} \\ \text{2 dynamical DOF} \end{array} \right. \end{cases}$$

$$K^{ij} = \psi^{-10}\left[(\tilde{\mathbb{L}}X)^{ij} + \tilde{Q}^{ij}\right] + \tfrac{1}{3}\gamma^{ij}K \begin{cases} \tilde{Q}^{ij} & : \left\langle \begin{array}{l} \tilde{\nabla}_j \tilde{Q}^{ij} = \tilde{Q}^i_i = 0 \\ \text{2 dynamical DOF} \end{array} \right. \\ X^i & : \left\langle \begin{array}{l} (\tilde{\mathbb{L}}X)^{ij} \equiv \tilde{\nabla}^i X^j + \tilde{\nabla}^j X^i - \tfrac{1}{3}\tilde{\gamma}^{ij}\tilde{\nabla}_k X^k \\ \text{3 constrained DOF} \end{array} \right. \\ K & : \text{1 temporal gauge DOF} \end{cases}$$

Specifying Initial Data

Freely specified degrees of freedom

$\tilde{\gamma}_{ij} \Leftarrow$ (2) initial dynamical ("wave") content; (3) initial spatial gauge choices

$\tilde{Q}^{ij} \Leftarrow$ (2) initial dynamical ("wave") content

$K \Leftarrow$ (1) initial temporal gauge choice

Freely specified *dynamical* gauge freedom

$\beta^i \Leftarrow$ how spatial coordinates evolve

$\alpha \Leftarrow$ how time coordinate evolves

Constrained degrees of freedom

$\psi \Leftarrow$ (1) Hamiltonian constraint

$$\tilde{\nabla}^2\psi - \tfrac{1}{8}\psi\tilde{R} - \tfrac{1}{12}\psi^5K^2 + \tfrac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = -2\pi\psi^5\rho$$

$X^i \Leftarrow$ (3) Momentum constraint

$$\tilde{\Delta}_{\mathbb{L}}X^i = \tfrac{2}{3}\psi^6\tilde{\nabla}^iK + 8\pi\psi^{10}j^i$$

$$\tilde{A}^{ij} \equiv (\tilde{\mathbb{L}}X)^{ij} + \tilde{Q}^{ij}$$

$$\tilde{\Delta}_{\mathbb{L}}X^i \equiv \tilde{\nabla}_j(\tilde{\mathbb{L}}X)^{ij}$$

Boundary conditions

The constraints form a set of 4 coupled nonlinear PDEs for (ψ, X^i) that require the specification of boundary conditions at spatial infinity and any interior boundaries.

"Traditional" Black-Hole Data

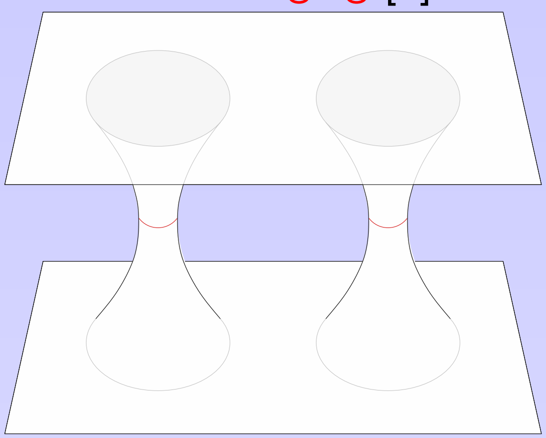
Conformal flatness and maximal slicing

$$\left.\begin{array}{l}\tilde{\gamma}_{ij} = f_{ij} \text{ (flat)} \\ \tilde{Q}^{ij} = 0 \\ K = 0\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\tilde{\Delta}_{\mathbb{L}} X^i = 0 \quad \Rightarrow \\ \tilde{\nabla}^2\psi + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = 0\end{array}\right.$$

Bowen-York solution[3]
Analytic solutions for $\tilde{A}^{ij}$

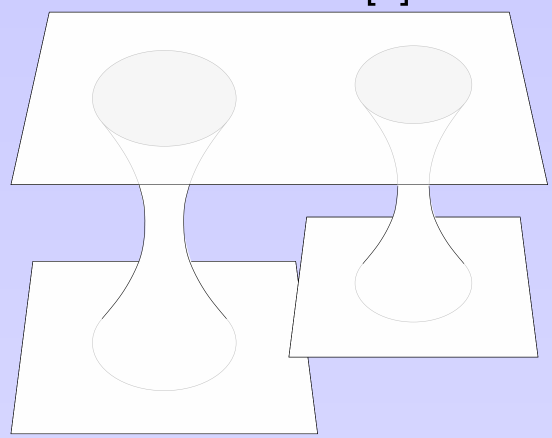
Three general solution schemes

Conformal Imaging-[7]



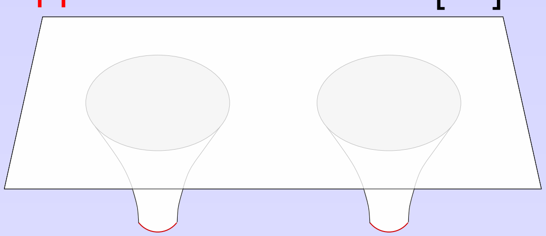
Inversion symmetry inner-BC

Puncture Method-[4]



No inner-BC: singular behavior factored out

Apparent Horizon BC-[12]



Apparent horizon inner-BC

All methods can produce very general configurations of multiple black holes, but are fundamentally limited by choices for $\tilde{\gamma}_{ij}$ and Bowen-York $\tilde{A}^{ij}$.

“Better” Black-Hole Data

What is wrong with “traditional” BH initial data?

- Results disagree with PN predictions for black holes in quasi-circular orbits.
- There is no control of the initial “wave” content.
- Spinning holes are not represented well.

How do we construct improved BH initial data?

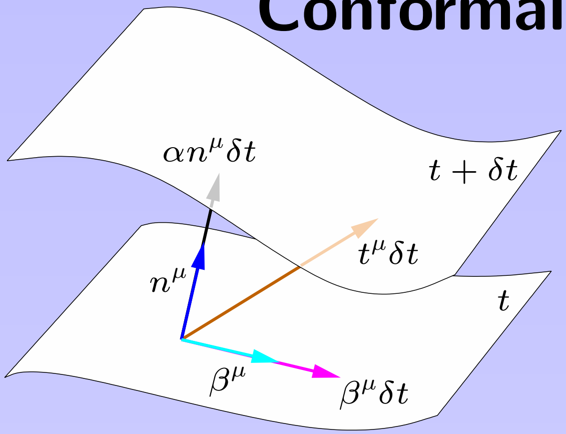
We must carefully choose the

- initial dynamical degrees of freedom [in $\tilde{\gamma}_{ij}$ and $\tilde{Q}^{ij}$]
- initial temporal and spatial gauge degrees of freedom [in $\tilde{\gamma}_{ij}$ and K]
- boundary conditions on the constrained degrees of freedom [in ψ and X^i]

so as to conform to the desired physical content of the initial data.

- For black holes in quasi-circular orbits, we can use the principle of *quasi-equilibrium* to guide our choices.
- Quasi-equilibrium is a *dynamical* concept and we can simplify our task by choosing a decomposition of the initial-data variables that has connections to dynamics.

Conformal Thin-Sandwich Decomposition[14]



$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$$

$$K^{ij} = \frac{\psi^{-10}}{2\tilde{\alpha}}\left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij}\right] + \tfrac{1}{3}\gamma^{ij}K \begin{cases} \tilde{u}_{ij} \equiv \partial_t \tilde{\gamma}_{ij} & (\tilde{u}^i_i = 0) \\ \tilde{\alpha} \equiv \psi^{-6}\alpha \end{cases}$$

Hamiltonian Const. $\tilde{\nabla}^2\psi - \tfrac{1}{8}\psi\tilde{R} - \tfrac{1}{12}\psi^5K^2 + \tfrac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = -2\pi\psi^5\rho$

Momentum Const. $\tilde{\Delta}_{\mathbb{L}}\beta^i - (\tilde{\mathbb{L}}\beta)^{ij}\tilde{\nabla}_j\tilde{\alpha} = \tfrac{4}{3}\tilde{\alpha}\psi^6\tilde{\nabla}^iK + \tilde{\alpha}\tilde{\nabla}_j\left(\tfrac{1}{\tilde{\alpha}}\tilde{u}^{ij}\right) + 16\pi\tilde{\alpha}\psi^{10}j^i$

$$\tilde{A}^{ij} \equiv \tfrac{1}{2\tilde{\alpha}}\left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij}\right]$$

Constrained vars : ψ and β^i

Freely specified : $\tilde{\gamma}_{ij}$, $\tilde{u}^{ij}$, K, and $\tilde{\alpha}$

$\tilde{u}^{ij}$ and β^i have a simple physical interpretation, unlike $\tilde{Q}^{ij}$ and X^i.

Quasi-equilibrium $\Rightarrow \begin{cases} \tilde{u}^{ij} = 0 \\ \partial_t K = 0 \text{ (Const. on } \alpha) \end{cases}$

Const. Tr(K) eqn. $\tilde{\nabla}^2(\alpha\psi) - \alpha\left[\tfrac{1}{8}\psi\tilde{R} + \tfrac{5}{12}\psi^5K^2 + \tfrac{7}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} + 2\pi\psi^5K(\rho + 2S)\right] = \psi^5\beta^i\tilde{\nabla}_iK$

Equations of Quasi-Equilibrium

Ham. & Mom. const.
eqns. from Conf. TS
$+$ Const. Tr(K) eqn.
$\Bigg\}\Rightarrow$ **Eqns. of Quasi-Equilibrium**

With $\tilde{\gamma}_{ij} = f_{ij}$, $\tilde{u}^{ij} = 0$, and $K = 0$, these equations have been widely used to construct binary neutron star initial data[1, 11, 2, 13].

Binary neutron star initial data require:

- boundary conditions at infinity compatible with asymptotic flatness and corotation.

$$\psi|_{\mathrm{r}\to\infty} = 1 \quad \beta^i|_{\mathrm{r}\to\infty} = \Omega\left(\frac{\partial}{\partial\phi}\right)^i \quad \alpha|_{\mathrm{r}\to\infty} = 1$$

- compatible solution of the equations of hydrostatic equilibrium. $(\Rightarrow \Omega)$

Binary black hole initial data require:

- a means for choosing the angular velocity of the orbit Ω.

⋆ *with excision*, inner boundary conditions are needed for ψ, β^i, and $\tilde{\alpha}$.

Gourgoulhon, Grandclément, & Bonazzola[9, 10]: Black-hole binaries with $\tilde{\gamma}_{ij} = f_{ij}$, $\tilde{u}^{ij} = 0$, $K = 0$, "inversion-symmetry", and "Killing-horizon" conditions on the excision boundaries.

"Solutions" require *constraint violating* regularity condition imposed on inner boundaries!

Constructing Regular Binary Black Hole QE ID

Why does the GGB approach have problems?

- Inversion-symmetry demands $\tilde{\alpha} = 0$ & $K = 0$ on the inner boundary.

- It is hard to move beyond $\tilde{\gamma}_{ij} = f_{ij}$.

$$\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}}\left[(\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij}\right]$$

How do we proceed?

- Find a method that allows for general choices of $\tilde{\gamma}_{ij}$ & K.

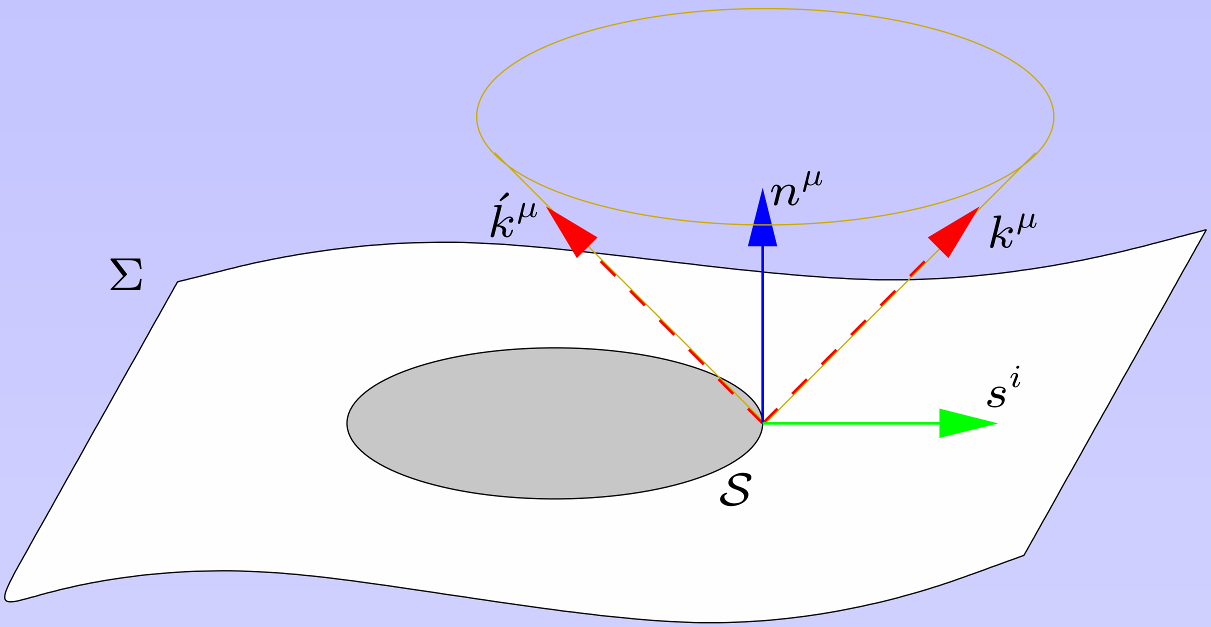
⋆ Eliminate dependence on inversion symmetry by letting the physical condition of quasi-equilibrium dictate the boundary conditions.

Approach

- Demand that the excision *(inner)* boundary be an *apparent horizon*.

- Demand that the apparent horizon be in quasi-equilibrium.

The Inner Boundary



$$s_i \equiv \frac{\bar{\nabla}_i\tau}{|\bar{\nabla}\tau|}$$

$$h_{ij} \equiv \gamma_{ij} - s_i s_j$$

$$k^\mu \equiv \tfrac{1}{\sqrt{2}}(n^\mu + s^\mu)$$

$$\acute{k}^\mu \equiv \tfrac{1}{\sqrt{2}}(n^\mu - s^\mu)$$

Extrinsic curvature of $\mathcal{S}$ embedded in spacetime

$$\Sigma_{\mu\nu} \equiv -\tfrac{1}{2}h^\alpha_\mu h^\beta_\nu \mathcal{L}_k g_{\alpha\beta}$$

$$\acute{\Sigma}_{\mu\nu} \equiv -\tfrac{1}{2}h^\alpha_\mu h^\beta_\nu \mathcal{L}_{\acute{k}} g_{\alpha\beta}$$

Extrinsic curvature of $\mathcal{S}$ embedded in Σ

$$H_{ij} \equiv -\tfrac{1}{2}h^k_i h^\ell_j \mathcal{L}_s \gamma_{k\ell}$$

$$\Sigma_{ij} = \tfrac{1}{\sqrt{2}}(J_{ij} + H_{ij})$$

$$\acute{\Sigma}_{ij} = \tfrac{1}{\sqrt{2}}(J_{ij} - H_{ij})$$

Projections of K_{ij} onto $\mathcal{S}$

$$J_{ij} \equiv h^k_i h^\ell_j K_{k\ell}$$

$$J_i \equiv h^k_i s^\ell K_{k\ell}$$

$$J \equiv h^{ij} J_{ij} = h^{ij} K_{ij}$$

Expansion of null rays

$$\theta \equiv h^{ij}\Sigma_{ij} = \tfrac{1}{\sqrt{2}}(J + H)$$

$$\acute{\theta} \equiv h^{ij}\acute{\Sigma}_{ij} = \tfrac{1}{\sqrt{2}}(J - H)$$

Shear of null rays

$$\sigma_{ij} \equiv \Sigma_{ij} - \tfrac{1}{2}h_{ij}\theta$$

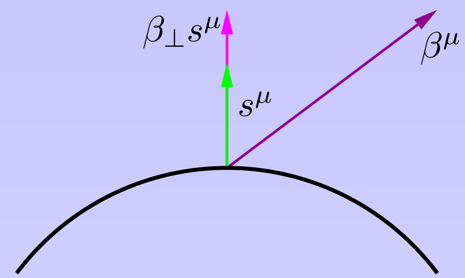
$$\acute{\sigma}_{ij} \equiv \acute{\Sigma}_{ij} - \tfrac{1}{2}h_{ij}\acute{\theta}$$

AH and QE Conditions on the Inner Boundary

The quasi-equilibrium inner boundary conditions start with the following assumptions:

1. The inner boundary $\mathcal{S}$ is a (MOTS):
   marginally outer-trapped surface

   $\longrightarrow \quad \theta = 0$

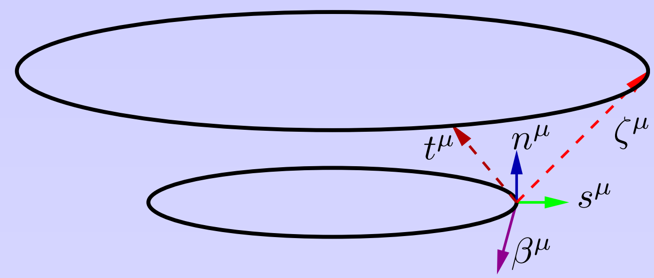


2. The inner boundary $\mathcal{S}$ doesn't move:

   $\longrightarrow \quad \mathcal{L}_\zeta\tau = 0$ and $\hat{\nabla}_i\mathcal{L}_\zeta\tau \equiv h_i^j\bar{\nabla}_j\mathcal{L}_\zeta\tau = 0$

   $$t^\mu = \alpha n^\mu + \beta^\mu \qquad \zeta^\mu \equiv \alpha n^\mu + \beta_\perp s^\mu$$

   $$\beta_\perp \equiv \beta^i s_i$$



3. The inner boundary $\mathcal{S}$ remains a MOTS[8]:

   $\longrightarrow \quad \mathcal{L}_\zeta\theta = 0$ **and** $\mathcal{L}_\zeta\acute{\theta} = 0$

4. The horizons are in quasi-equilibrium:

   $\longrightarrow \quad \sigma_{ij} = 0$ and no matter is on $\mathcal{S}$

Evolution of the Expansions

$$
\begin{aligned}
\mathcal{L}_\zeta \theta &= \tfrac{1}{\sqrt{2}}\left[\theta(\theta + \tfrac{1}{2}\acute{\theta} - \tfrac{1}{\sqrt{2}}K) + \mathcal{E}\right](\beta_\perp + \alpha) \\
&\quad + \tfrac{1}{\sqrt{2}}\left[\theta(\tfrac{1}{2}\theta - \tfrac{1}{2}\acute{\theta} - \tfrac{1}{\sqrt{2}}K) + \mathcal{D} + 8\pi T_{\mu\nu}k^\mu \acute{k}^\nu\right](\beta_\perp - \alpha) \\
&\quad + \theta s^i \bar{\nabla}_i \alpha, \\
\mathcal{L}_\zeta \acute{\theta} &= -\tfrac{1}{\sqrt{2}}\left[\acute{\theta}(\acute{\theta} + \tfrac{1}{2}\theta - \tfrac{1}{\sqrt{2}}K) + \acute{\mathcal{E}}\right](\beta_\perp - \alpha) \\
&\quad - \tfrac{1}{\sqrt{2}}\left[\acute{\theta}(\tfrac{1}{2}\acute{\theta} - \tfrac{1}{2}\theta - \tfrac{1}{\sqrt{2}}K) + \acute{\mathcal{D}} + 8\pi T_{\mu\nu}k^\mu \acute{k}^\nu\right](\beta_\perp + \alpha) \\
&\quad - \acute{\theta} s^i \bar{\nabla}_i \alpha,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D} &\equiv h^{ij}(\hat{\nabla}_i + J_i)(\hat{\nabla}_j + J_j) - \tfrac{1}{2}\hat{R} \\
\acute{\mathcal{D}} &\equiv h^{ij}(\hat{\nabla}_i - J_i)(\hat{\nabla}_j - J_j) - \tfrac{1}{2}\hat{R} \\
\mathcal{E} &\equiv \sigma_{ij}\sigma^{ij} + 8\pi T_{\mu\nu}k^\mu k^\nu \\
\acute{\mathcal{E}} &\equiv \acute{\sigma}_{ij}\acute{\sigma}^{ij} + 8\pi T_{\mu\nu}\acute{k}^\mu \acute{k}^\nu
\end{aligned}
$$

Incorporates the constraint and evolution equations of GR, the Gauss–Codazzi–Ricci equations governing the embedding of $\mathcal{S}$ in the spatial hypersurface, and the demand that $\mathcal{S}$ remain at a constant coordinate location. *These equations incorporate no assumption of quasi-equilibrium.*

"Red" terms vanish because we demand $\mathcal{S}$ be a MOTS, remain a MOTS, or because we demand the horizon to be in equilibrium.

AH/Quasi-Equilibrium Boundary Conditions

$$\theta = 0$$

$$0 = \mathcal{D}(\beta_\perp - \alpha),$$

$$\acute{\theta} s^i \bar{\nabla}_i \alpha = -\frac{1}{\sqrt{2}}\left[\acute{\theta}(\acute{\theta} - \frac{1}{\sqrt{2}}K) + \acute{\sigma}_{ij}\acute{\sigma}^{ij}\right](\beta_\perp - \alpha) - \frac{1}{\sqrt{2}}\left[\acute{\theta}(\tfrac{1}{2}\acute{\theta} - \frac{1}{\sqrt{2}}K) + \acute{\mathcal{D}}\right](\beta_\perp + \alpha).$$

$\Rightarrow$

$$\tilde{s}^k \tilde{\nabla}_k \ln\psi = -\tfrac{1}{4}(\tilde{h}^{ij}\tilde{\nabla}_i \tilde{s}_j - \psi^2 J)$$

$$\beta^i = \alpha\psi^{-2}\tilde{s}^i + \beta^i_{\parallel}$$

$$J\tilde{s}^i\tilde{\nabla}_i\alpha = -\psi^2(J^2 - JK + \tilde{\mathcal{D}})\alpha$$

$$h_{ij} \equiv \psi^4 \tilde{h}_{ij}$$

$$s^i \equiv \psi^{-2}\tilde{s}^i$$

$$\beta^i_{\parallel} s_i = 0$$

$$\tilde{\mathcal{D}} \equiv \psi^{-4}[\tilde{h}^{ij}(\breve{\nabla}_i - J_i)(\breve{\nabla}_j - J_j) - \tfrac{1}{2}\breve{R} + 2\breve{\nabla}^2 \ln\psi]$$

[$\breve{\nabla}$ & $\breve{R}$ are compatible with $\tilde{h}_{ij}$]

The conditions of quasi-equilibrium yield boundary conditions for 3 of the 5 constrained variables (ψ, α, $\beta_\perp$). The remaining two conditions are contained in the definition of $\beta^i_{\parallel}$. This freedom is necessary to prescribe the spin of the black hole.

Defining the Spin of the Black Hole

The spin parameters $\beta^i_{\|}$ can be defined by demanding that the MOTS be a *Killing horizon*. The time vector associated with quasi-equilibrium in the corotating frame must be null, forming the null generators of the horizon.

$$k^\mu \propto (n^\mu + s^\mu) \implies k^\mu = \left[1, \alpha s^i - \beta^i\right]$$

This vector k^μ is null for any choice of α & β^i.

In the frame where a black hole is not spinning, the null time vector has components $t^\mu = [1, \vec{0}]$.

Corotating Holes

Corotating holes are at rest in the corotating frame, where we must pose boundary conditions. So,

$$k^\mu = \left[1, \alpha s^i - \beta^i\right] = [1, \vec{0}]$$

Thus we find

$$\beta^i = \alpha s^i \;\Rightarrow\; \beta^i_{\|} = 0$$

Irrotational Holes

Irrotational holes are at rest in the inertial frame. With the time vectors in the inertial and corotating frames related by

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} + \Omega \frac{\partial}{\partial \phi}$$

$$k^\mu = \left[1, \alpha s^i - \beta^i\right] = [1, -\Omega(\partial/\partial\phi)^i]$$

Thus we find

$$\beta^i = \alpha s^i + \Omega \left(\frac{\partial}{\partial \phi}\right)^i \Rightarrow \beta^i_{\|} = \Omega \left(\frac{\partial}{\partial \phi}\right)^i$$

Summary of QE Formalism

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \qquad K^{ij} = \psi^{-10}\tilde{A}^{ij} + \tfrac{1}{3}\gamma^{ij}K \qquad \tilde{A}^{ij} = \frac{\psi^6}{2\alpha}(\tilde{\mathbb{L}}\beta)^{ij}$$

$\partial_t \tilde{\gamma}_{ij} = 0$

$$\begin{aligned}
\tilde{\nabla}^2\psi - \tfrac{1}{8}\psi\tilde{R} - \tfrac{1}{12}\psi^5K^2 + \tfrac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} &= 0 \\
\tilde{\Delta}_{\mathbb{L}}\beta^i - (\tilde{\mathbb{L}}\beta)^{ij}\tilde{\nabla}_j \ln\alpha\psi^{-6} &= \tfrac{4}{3}\alpha\tilde{\nabla}^iK \\
\tilde{\nabla}^2(\alpha\psi) - (\alpha\psi)\Big[\tfrac{1}{8}\tilde{R} + \tfrac{5}{12}\psi^4K^2 + \tfrac{7}{8}\psi^{-8}\tilde{A}_{ij}\tilde{A}^{ij}\Big] &= \psi^5\beta^i\tilde{\nabla}_iK
\end{aligned}$$

$\partial_t K = 0$

$$\tilde{s}^k\tilde{\nabla}_k \ln\psi|_{\mathcal{S}} = -\tfrac{1}{4}(\tilde{h}^{ij}\tilde{\nabla}_i\tilde{s}_j - \psi^2J)|_{\mathcal{S}}$$

$\theta = 0$

$$\beta^i|_{\mathcal{S}} = \begin{cases} \alpha\psi^{-2}\tilde{s}^i|_{\mathcal{S}} & \text{corotation} \\ \alpha\psi^{-2}\tilde{s}^i|_{\mathcal{S}} + \Omega\tilde{h}^i_j\left(\frac{\partial}{\partial\phi}\right)^j\Big|_{\mathcal{S}} & \text{irrotation} \end{cases}$$

$\mathcal{L}_\zeta\theta = 0$

$\sigma_{ij} = 0$

$$J\tilde{s}^i\tilde{\nabla}_i\alpha|_{\mathcal{S}} = -\psi^2(J^2 - JK + \tilde{\mathcal{D}})\alpha|_{\mathcal{S}}$$

$\mathcal{L}_\zeta\acute{\theta} = 0$

$$\begin{aligned}
\psi|_{\mathrm{r}\to\infty} &= 1 \\
\beta^i|_{\mathrm{r}\to\infty} &= \Omega\left(\frac{\partial}{\partial\phi}\right)^i \\
\alpha|_{\mathrm{r}\to\infty} &= 1
\end{aligned}$$

The only remaining freedom in the system is the choice of the orbital angular velocity, the initial spatial and temporal gauge, and the initial dynamical("wave") content found in Ω, $\tilde{\gamma}_{ij}$ and K.

The Orbital Angular Velocity

- For a given choice of $\tilde{\gamma}_{ij}$ and K, we are still left with a family of solutions parameterized by the orbital angular velocity Ω.
- Except for the case of a single spinning black hole, it is not reasonable to expect more than one value of Ω to correspond to a system in quasi-equilibrium.

GGB[9, 10] have suggested a way to pick the quasi-equilibrium value of Ω:

Ω is chosen as the value for which the ADM energy $E_{\rm ADM}$ equals the Komar mass $M_{\rm K}$.

Komar mass	$M_{\rm K} = \frac{1}{4\pi}\oint_\infty \gamma^{ij}(\bar\nabla_i\alpha - \beta^k K_{ik})\mathrm{d}^2S_j$	Acceptable definition of the mass *only for stationary spacetimes*.
ADM energy	$E_{\rm ADM} = \frac{1}{16\pi}\oint_\infty \gamma^{ij}\bar\nabla_k(\mathcal{G}^k_i - \delta^k_i\mathcal{G})\mathrm{d}^2S_j$	Acceptable definition of the mass *for arbitrary spacetimes*. $\mathcal{G}_{ij} \equiv \gamma_{ij} - f_{ij}$

Do the AH/QE BCs Yield a Well Posed System?

Single Black Hole tests: **Implementation and results due to H. Pfeiffer**

- $\tilde{\gamma}_{ij}$ and K from Kerr-Schild:
  - AH/QE BCs seem ill-conditioned with slow/no nonlinear convergence.
  - Replacing the BC on either α or $\beta_\perp$ with the proper Dirichlet data yields good convergence.
  - Replacing the BC on either α or $\beta_\perp$ with the **wrong** Dirichlet data yields good convergence.
  - Solving with Dirichlet BC replacing one of the BCs yields a solution that:
    - ⋆ obeys the full AH/QE BCs
    - ⋆ has $\partial_t \psi = 0$

    (if the outer boundary is at ∞)

- $\tilde{\gamma}_{ij} = f_{ij}$ and $K = 1/r^2$ or 0
  - Solving with Dirichlet BC replacing one of the BCs yields a solution that:
    - ⋆ obeys the full AH/QE BCs
    - ⋆ has $\partial_t \psi = 0$

    (if the outer boundary is at ∞)

Fixing the Length Scale

These results suggest that, at lest in spherical symmetry, the AH/QE BC's yield a one-parameter family of solutions. How do we fix a unique solution?

- Modify the BC on ψ so that it also fixes the average value of ψ:
  - AH/QE BCs + $\bar{\psi}$ converges *if the initial guess is "good"*.
  - With a poor initial guess, the solution gets caught in a local minimum.
- Modify the BC on α so that it also fixes the average value of α:
  - AH/QE BCs + $\bar{\alpha}$ converges *if the initial guess is "good"*.
  - With a poor initial guess, the solution gets caught in a local minimum.

Convergence problems are fixed for general, non-spherical cases if $\bar{\psi}$ or $\bar{\alpha}$ are fixed **and** one term in QE BC on α is changed:

- $\tilde{h}^{ij}\breve{\nabla}_i J_j \to 0$
- $\tilde{h}^{ij}\breve{\nabla}_i J_j \to -\tilde{h}^{ij}\breve{\nabla}_i J_j$

We are still trying to determine if this is a bug in the code or a fundamental problem

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