

# Black-Hole Binary Initial Data: Getting the Spin Right

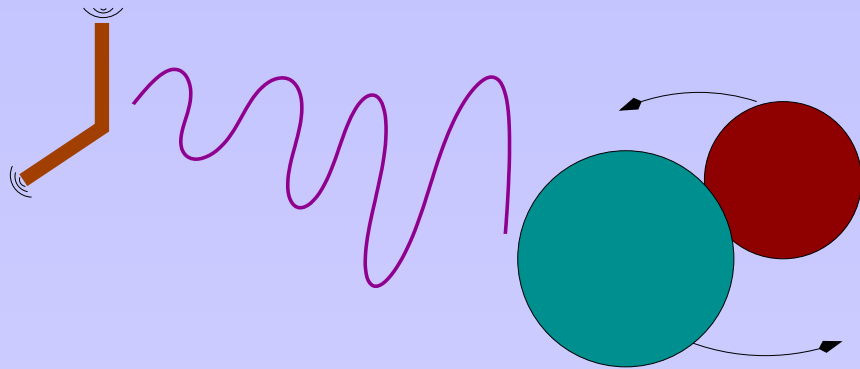
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Collaborators: Harald Pfeiffer<sup>[7]</sup> (Caltech), Jason D. Grigsby (WFU), & Matthew Caudill (WFU)

# Motivation

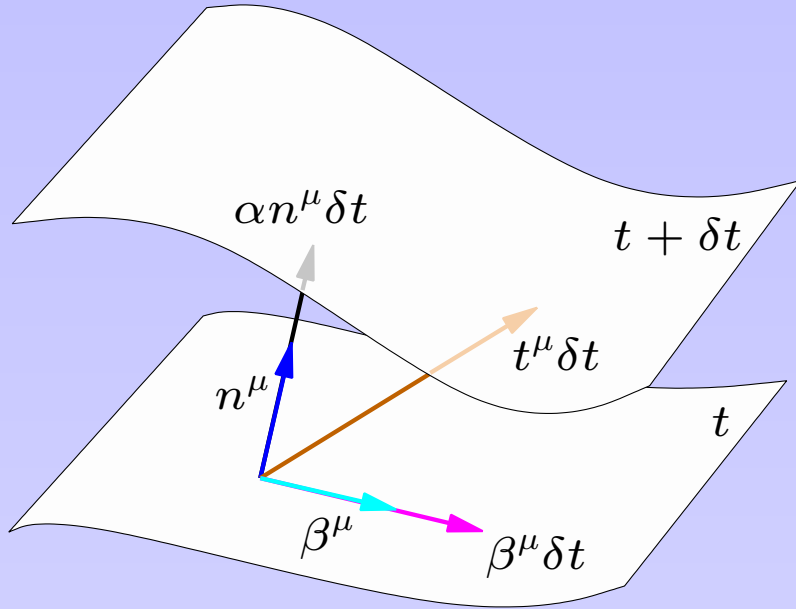


- Black hole binaries are among the most likely sources for early detection with LIGO, VIRGO, GEO, TAMA, . . .
- Available computed waveforms should increase chance of detecting collision events.

## Quasiequilibrium Binary Data

- General Relativity doesn't permit *true* equilibrium for astrophysical binary systems.
- When the bodies are sufficiently far apart, the timescale for orbital decay is much larger than the orbital period.
- If the orbit is nearly circular (*quasi-circular*) then there is a *corotating reference frame* in which the binary appears to be at rest.
- ★ Quasiequilibrium gives us a *physical condition* to guide us in fixing boundary conditions and data that is not otherwise constrained.

# The 3 + 1 Decomposition



Lapse :  $\alpha$

Spatial metric :  $\gamma_{ij}$

Shift vector :  $\beta^i$

Extrinsic Curvature :  $K_{ij}$

Time vector :  $t^\mu = \alpha n^\mu + \beta^\mu$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \quad K_{\mu\nu} = -\frac{1}{2}\gamma_\mu^\alpha \gamma_\nu^\beta \mathcal{L}_n g_{\alpha\beta}$$

## Constraint equations

$$\bar{R} + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

$$\bar{\nabla}_j (K^{ij} - \gamma^{ij}K) = 8\pi j^i$$

$$S_{\mu\nu} \equiv \gamma_\mu^\alpha \gamma_\nu^\beta T_{\alpha\beta}$$

$$j_\mu \equiv -\gamma_\mu^\nu n^\alpha T_{\nu\alpha}$$

$$\rho \equiv n^\mu n^\nu T_{\mu\nu}$$

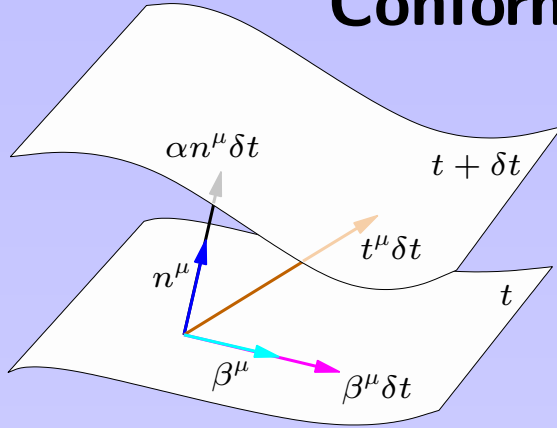
$$T_{\mu\nu} = S_{\mu\nu} + 2n_{(\mu}j_{\nu)} + n_\mu n_\nu \rho$$

## Evolution equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i$$

$$\begin{aligned} \partial_t K_{ij} = & -\bar{\nabla}_i \bar{\nabla}_j \alpha + \alpha \left[ \bar{R}_{ij} - 2K_{il}K_j^l + KK_{ij} \right. \\ & \left. - 8\pi S_{ij} + 4\pi \gamma_{ij}(S - \rho) \right] \\ & + \beta^\ell \bar{\nabla}_\ell K_{ij} + K_{il} \bar{\nabla}_j \beta^\ell + K_{jl} \bar{\nabla}_i \beta^\ell \end{aligned}$$

# Conformal Thin-Sandwich Decomposition



$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}$$

$$K^{ij} = \frac{\psi^{-10}}{2\tilde{\alpha}} \left[ (\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right] + \frac{1}{3}\psi^{-4}\tilde{\gamma}^{ij}K$$

**Hamiltonian Const.**  $\tilde{\nabla}^2\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{12}\psi^5K^2 + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = -2\pi\psi^5\rho$

**Momentum Const.**  $\tilde{\nabla}_j(\tilde{\mathbb{L}}\beta)^{ij} - (\tilde{\mathbb{L}}\beta)^{ij}\tilde{\nabla}_j\tilde{\alpha} = \frac{4}{3}\tilde{\alpha}\psi^6\tilde{\nabla}^iK + \tilde{\alpha}\tilde{\nabla}_j\left(\frac{1}{\tilde{\alpha}}\tilde{u}^{ij}\right) + 16\pi\tilde{\alpha}\psi^{10}j^i$

**Const. Tr(K) eqn.**  $\tilde{\nabla}^2(\psi^7\tilde{\alpha}) - (\psi^7\tilde{\alpha})\left[\frac{1}{8}\psi\tilde{R} + \frac{5}{12}\psi^5K^2 + \frac{7}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} - \psi^5\beta^i\tilde{\nabla}_iK\right]$   
 $= -2\pi\psi^5K(\rho + 2S) - \psi^5\partial_tK$

$$\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left[ (\tilde{\mathbb{L}}\beta)^{ij} - \tilde{u}^{ij} \right]$$

**Constrained vars :**  $\psi$ ,  $\beta^i$ , and  $\tilde{\alpha} \equiv \psi^{-6}\alpha$

**Freely specified :**  $\tilde{\gamma}_{ij}$   $\tilde{u}^{ij} \equiv -\partial_t\tilde{\gamma}^{ij}$   
 $K$  *and*  $\partial_tK$

$$\text{Quasiequilibrium} \Rightarrow \begin{cases} \partial_t\tilde{\gamma}^{ij} = 0 \\ \partial_tK = 0 \end{cases}$$

# Equations of Quasiequilibrium

$$\left. \begin{array}{l} \text{Ham. \& Mom. const.} \\ \text{eqns., \& Const Tr}(K) \\ \text{eqn. from Conf. TS} \\ + \tilde{u}^{ij} = \partial_t K = 0 \end{array} \right\} \Rightarrow \text{Eqns. of} \\ \text{Quasiequilibrium}$$

With  $\tilde{\gamma}_{ij} = f_{ij}$  and  $K = 0$ , these equations have been widely used to construct binary neutron star initial data [3, 12, 4, 15].

Binary neutron star initial data require:

- boundary conditions at infinity compatible with asymptotic flatness and corotation.

$$\psi|_{r \rightarrow \infty} = 1 \quad \beta^i|_{r \rightarrow \infty} = \Omega_0 \left( \frac{\partial}{\partial \phi} \right)^i \quad \alpha|_{r \rightarrow \infty} = 1$$

- compatible solution of the equations of hydrostatic equilibrium. ( $\Rightarrow \Omega_0$ )

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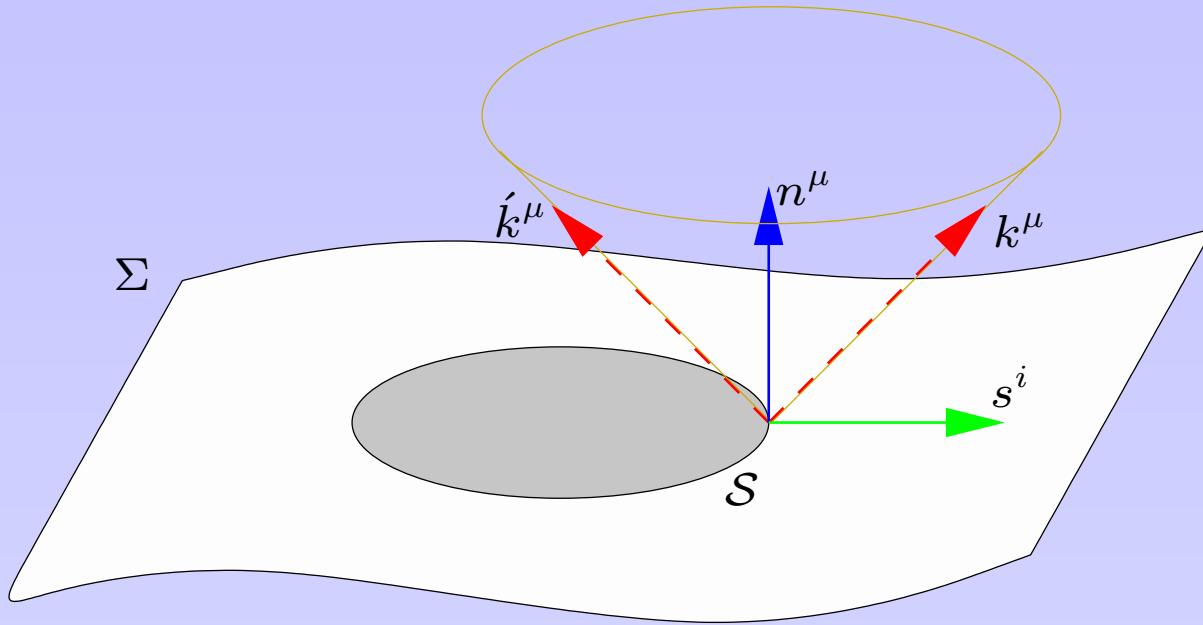
Binary black hole initial data require:

- ★ a means for choosing the angular velocity of the orbit  $\Omega_0$ .
- ★ *with excision*, inner boundary conditions are needed for  $\psi$ ,  $\beta^i$ , and  $\tilde{\alpha}$ .

Gourgoulhon, Grandclément, & Bonazzola[10, 11]: Black-hole binaries with  $\tilde{\gamma}_{ij} = f_{ij}$  &  $K = 0$ , “inversion-symmetry”, and “Killing-horizon” conditions on the excision boundaries.

“Solutions” require *constraint violating* regularity condition imposed on inner boundaries!

# The Inner Boundary



$$s_i \equiv \frac{\bar{\nabla}_i \tau}{|\bar{\nabla} \tau|}$$

$$h_{ij} \equiv \gamma_{ij} - s_i s_j$$

$$k^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu + s^\mu)$$

$$\hat{k}^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu - s^\mu)$$

Extrinsic curvature of  $S$  embedded in spacetime

$$\Sigma_{\mu\nu} \equiv \frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_k g_{\alpha\beta}$$

$$\hat{\Sigma}_{\mu\nu} \equiv \frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_{\hat{k}} g_{\alpha\beta}$$

Extrinsic curvature of  $S$  embedded in  $\Sigma$

$$H_{ij} \equiv \frac{1}{2} h_i^k h_j^\ell \mathcal{L}_s \gamma_{kl}$$

$$\Sigma_{ij} = \frac{1}{\sqrt{2}} (H_{ij} - J_{ij})$$

$$\hat{\Sigma}_{ij} = -\frac{1}{\sqrt{2}} (H_{ij} + J_{ij})$$

Projections of  $K_{ij}$  onto  $S$

$$J_{ij} \equiv h_i^k h_j^\ell K_{kl}$$

$$J_i \equiv h_i^k s^\ell K_{kl}$$

$$J \equiv h^{ij} J_{ij} = h^{ij} K_{ij}$$

Expansion of null rays

$$\theta \equiv h^{ij} \Sigma_{ij} = \frac{1}{\sqrt{2}} (H - J)$$

$$\hat{\theta} \equiv h^{ij} \hat{\Sigma}_{ij} = -\frac{1}{\sqrt{2}} (H + J)$$

Shear of null rays

$$\sigma_{ij} \equiv \Sigma_{ij} - \frac{1}{2} h_{ij} \theta$$

$$\hat{\sigma}_{ij} \equiv \hat{\Sigma}_{ij} - \frac{1}{2} h_{ij} \hat{\theta}$$

# AH and QE Conditions on the Inner Boundary

The quasiequilibrium inner boundary conditions start with the following assumptions:

1. The inner boundary  $\mathcal{S}$  is a (MOTS):  
marginally outer-trapped surface

$$\rightarrow \theta = 0$$

2. The horizons are in quasiequilibrium:

$$\rightarrow \sigma_{ij} = 0 \text{ and no matter is on } \mathcal{S}$$

Raychaudhuri's equation implies that MOTS initially evolves along  $k^\mu$ .

$$\mathcal{L}_k \theta = \frac{1}{2} \theta^2 - \sigma_{\mu\nu} \sigma^{\mu\nu} - \omega_{\mu\nu} \omega^{\mu\nu} - R_{\mu\nu} k^\mu k^\nu = 0$$

3. The time evolution vector lies in outgoing null surface through  $\mathcal{S}$ :

$$\rightarrow t^\mu k_\mu|_{\mathcal{S}} = 0$$

$$\left. \begin{aligned} k^\mu &\equiv \frac{1}{\sqrt{2}}(n^\mu + s^\mu) \\ t^\mu &= \alpha n^\mu + \beta^\mu \end{aligned} \right\} \Rightarrow \alpha|_{\mathcal{S}} = \beta^i s_i|_{\mathcal{S}} \equiv \beta_\perp|_{\mathcal{S}}$$

# AH/Quasiequilibrium Boundary Conditions

$$\theta = \frac{\psi^{-2}}{\sqrt{2}} \left[ \tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j + 4\tilde{s}^k \tilde{\nabla}_k \ln \psi - \psi^2 J \right]$$

$$\begin{aligned} \sigma_{ij} = & \frac{1}{\sqrt{2}} (H_{ij} - \frac{1}{2} h_{ij} H) \left( 1 - \frac{\beta_{\perp}}{\alpha} \right) \\ & - \frac{1}{\sqrt{2}} \frac{\psi^4}{\alpha} \left\{ \tilde{D}_{(i} \beta_{\parallel j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \beta_{\parallel}^k - \frac{1}{2} [\tilde{h}_{ik} \tilde{h}_{jl} \tilde{u}^{kl} - \frac{1}{2} \tilde{h}_{ij} \tilde{h}_{kl} \tilde{u}^{kl}] \right\} \end{aligned}$$

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$$\tilde{s}^k \tilde{\nabla}_k \ln \psi = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J)$$

$$\beta^i = \alpha \psi^{-2} \tilde{s}^i + \beta_{\parallel}^i$$

$$0 = \tilde{D}_{(i} \beta_{\parallel j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \beta_{\parallel}^k$$

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The boundary condition on the lapse is not fixed by QE assumptions. It has been shown to be associated with the initial temporal gauge choice and is taken to be freely specifiable.

# Defining the Spin of the Black Hole

The spin parameters  $\beta_{\parallel}^i$  can be defined by demanding that the time vector associated with quasiequilibrium in the corotating frame must be null, forming the null generators of the horizon.

$$k^{\mu} \propto (n^{\mu} + s^{\mu}) \implies k^{\mu} = [1, \alpha s^i - \beta^i]$$

*This vector  $k^{\mu}$  is null for any choice of  $\alpha$  &  $\beta^i$ .*

In the frame where a black hole is not spinning, the null time vector has components  $t^{\mu} = [1, \vec{0}]$ .

## Corotating Holes

Corotating holes are at rest in the corotating frame, where we must pose boundary conditions. So,

$$k^{\mu} = [1, \alpha s^i - \beta^i] = [1, \vec{0}]$$

Thus we find

$$\beta^i = \alpha s^i \implies \beta_{\parallel}^i = 0$$

## Irrotational Holes

Irrotational holes are at rest in the inertial frame. With the time vectors in the inertial and corotating frames related by

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} + \Omega_0 \frac{\partial}{\partial \phi}$$

$$k^{\mu} = [1, \alpha s^i - \beta^i] = [1, -\Omega_0 (\partial/\partial \phi)^i]$$

Thus we find

$$\beta^i = \alpha s^i + \Omega_0 \left( \frac{\partial}{\partial \phi} \right)^i \implies \beta_{\parallel}^i = \Omega_0 \xi^i$$

$$\xi^i \approx \left( \frac{\partial}{\partial \phi} \right)^i \quad \& \quad \tilde{D}_{(i} \xi_{j)} - \frac{1}{2} \tilde{h}_{ij} \tilde{D}_k \xi^k = 0$$

# The Lapse BC & QE

So far, nothing has fixed a boundary condition on the lapse  $\alpha$ . One possibility[9] is to recall that  $\theta\acute{\theta}$  is a Lorentz invariant and so to consider  $\mathcal{L}_\zeta\theta\acute{\theta} = 0$  as a quasiequilibrium condition.

$$\mathcal{L}_\zeta\theta\acute{\theta} = 0 \quad \Rightarrow \quad J\tilde{s}^i\tilde{\nabla}_i\alpha = -\psi^2(J^2 - JK + \tilde{\mathcal{D}})\alpha$$

$$\tilde{\mathcal{D}} \equiv \psi^{-4}[\tilde{h}^{ij}(\tilde{D}_i - J_i)(\tilde{D}_j - J_j) - \frac{1}{2}\tilde{R} + 2\tilde{D}^2 \ln \psi]$$

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This conditions is satisfied for stationary solutions, but seems to be degenerate with the other QE boundary conditions. To see this, note that the stationary maximal slicings of Schwarzschild form a 1-parameter family:

$$ds^2 = \frac{dR^2}{1 - \frac{2M}{R} + \frac{C^2}{R^4}} + R^2 d^2\Omega$$

$$\alpha = \sqrt{1 - \frac{2M}{R} + \frac{C^2}{R^4}}$$

$$\beta^R = \frac{C}{R^2} \sqrt{1 - \frac{2M}{R} + \frac{C^2}{R^4}}$$

$$K_j^i = \frac{C}{R^3} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha|_s = \frac{C}{4M^2}$$

# The Orbital Angular Velocity

- For a given choice of the Lapse BC,  $\tilde{\gamma}_{ij}$  and  $K$ , we are still left with a family of solutions parameterized by the orbital angular velocity  $\Omega_0$ .
- Except for the case of a single spinning black hole, it is not reasonable to expect more than one value of  $\Omega_0$  to correspond to a system in quasiequilibrium.

GGB[10, 11] have suggested a way to pick the quasiequilibrium value of  $\Omega_0$ :

$\Omega_0$  is chosen as the value for which the ADM energy  $E_{\text{ADM}}$  equals the Komar mass  $M_{\text{K}}$ .

Komar  
mass

$$M_{\text{K}} = \frac{1}{4\pi} \oint_{\infty} \gamma^{ij} (\bar{\nabla}_i \alpha - \beta^k K_{ik}) d^2 S_j$$

Acceptable definition of the mass  
*only for stationary spacetimes.*

ADM  
energy

$$E_{\text{ADM}} = \frac{1}{16\pi} \oint_{\infty} \gamma^{ij} \bar{\nabla}_k (\mathcal{G}_i^k - \delta_i^k \mathcal{G}) d^2 S_j$$

Acceptable definition of the mass  
*for arbitrary spacetimes.*

$$\mathcal{G}_{ij} \equiv \gamma_{ij} - f_{ij}$$

# Summary of QE Formalism

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad K^{ij} = \psi^{-10} \tilde{A}^{ij} + \frac{1}{3} \gamma^{ij} K \quad \tilde{A}^{ij} = \frac{\psi^6}{2\alpha} (\tilde{\mathbb{L}}\beta)^{ij} \quad \partial_t \tilde{\gamma}_{ij} = 0$$

$$\tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0$$

$$\tilde{\nabla}_j (\tilde{\mathbb{L}}\beta)^{ij} - (\tilde{\mathbb{L}}\beta)^{ij} \tilde{\nabla}_j \ln \alpha \psi^{-6} = \frac{4}{3} \alpha \tilde{\nabla}^i K$$

$$\tilde{\nabla}^2 (\alpha \psi) - (\alpha \psi) \left[ \frac{1}{8} \tilde{R} + \frac{5}{12} \psi^4 K^2 + \frac{7}{8} \psi^{-8} A_{ij} A^{ij} \right] = \psi^5 \beta^i \tilde{\nabla}_i K \quad \partial_t K = 0$$

$$\tilde{s}^k \tilde{\nabla}_k \ln \psi|_S = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J)|_S \quad \theta = 0$$

$$\beta^i|_S = \begin{cases} \alpha \psi^{-2} \tilde{s}^i|_S & \text{corotation} \\ \alpha \psi^{-2} \tilde{s}^i|_S + \Omega_0 \xi^i|_S & \text{irrotation} \end{cases} \quad \begin{matrix} \sigma_{ij} = 0 \\ t^\mu k_\mu = 0 \end{matrix}$$

$\alpha|_S = \text{unspecified by QE}$

$$\psi|_{r \rightarrow \infty} = 1$$

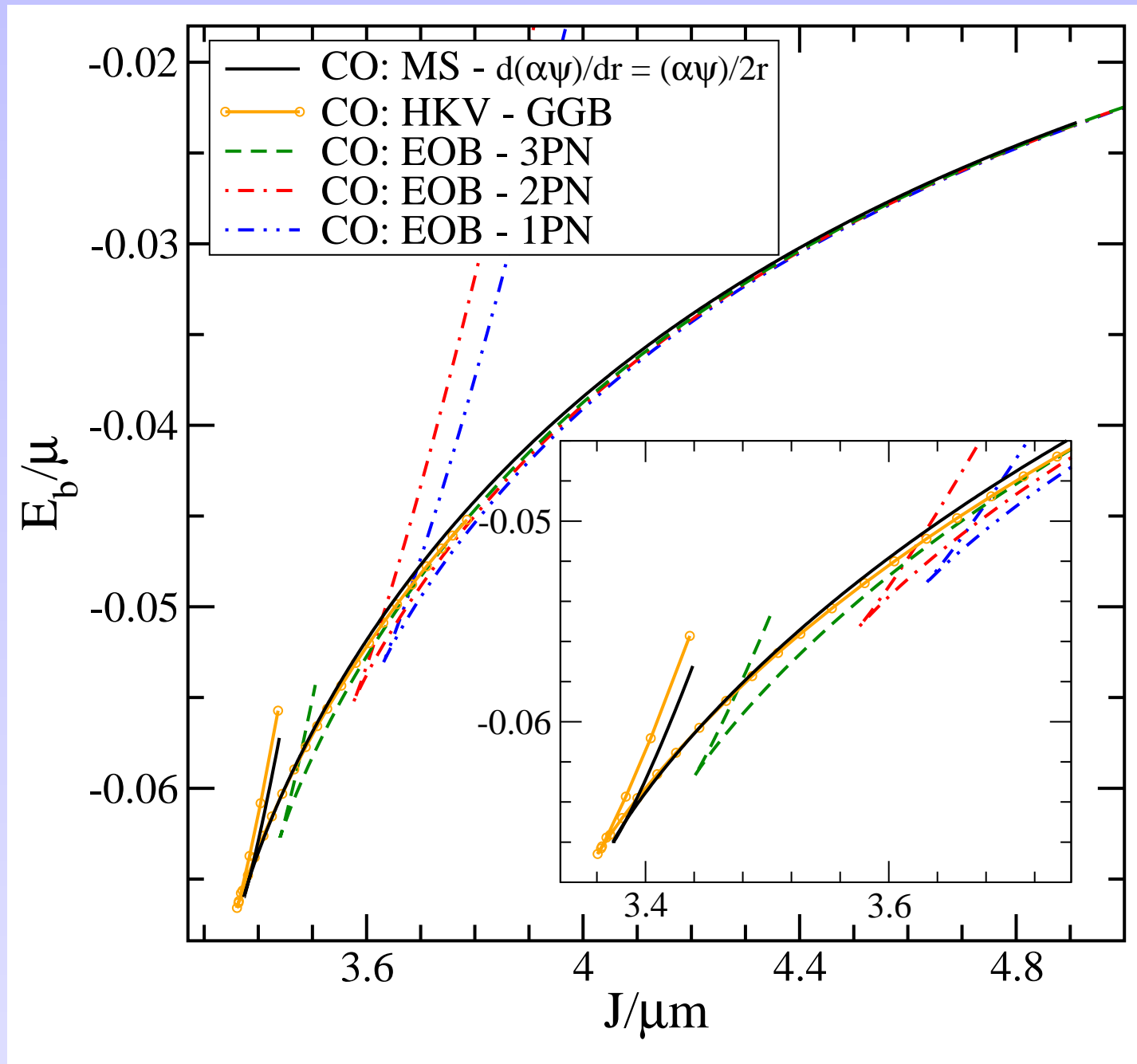
$$\beta^i|_{r \rightarrow \infty} = \Omega_0 \left( \frac{\partial}{\partial \phi} \right)^i$$

$$\alpha|_{r \rightarrow \infty} = 1$$

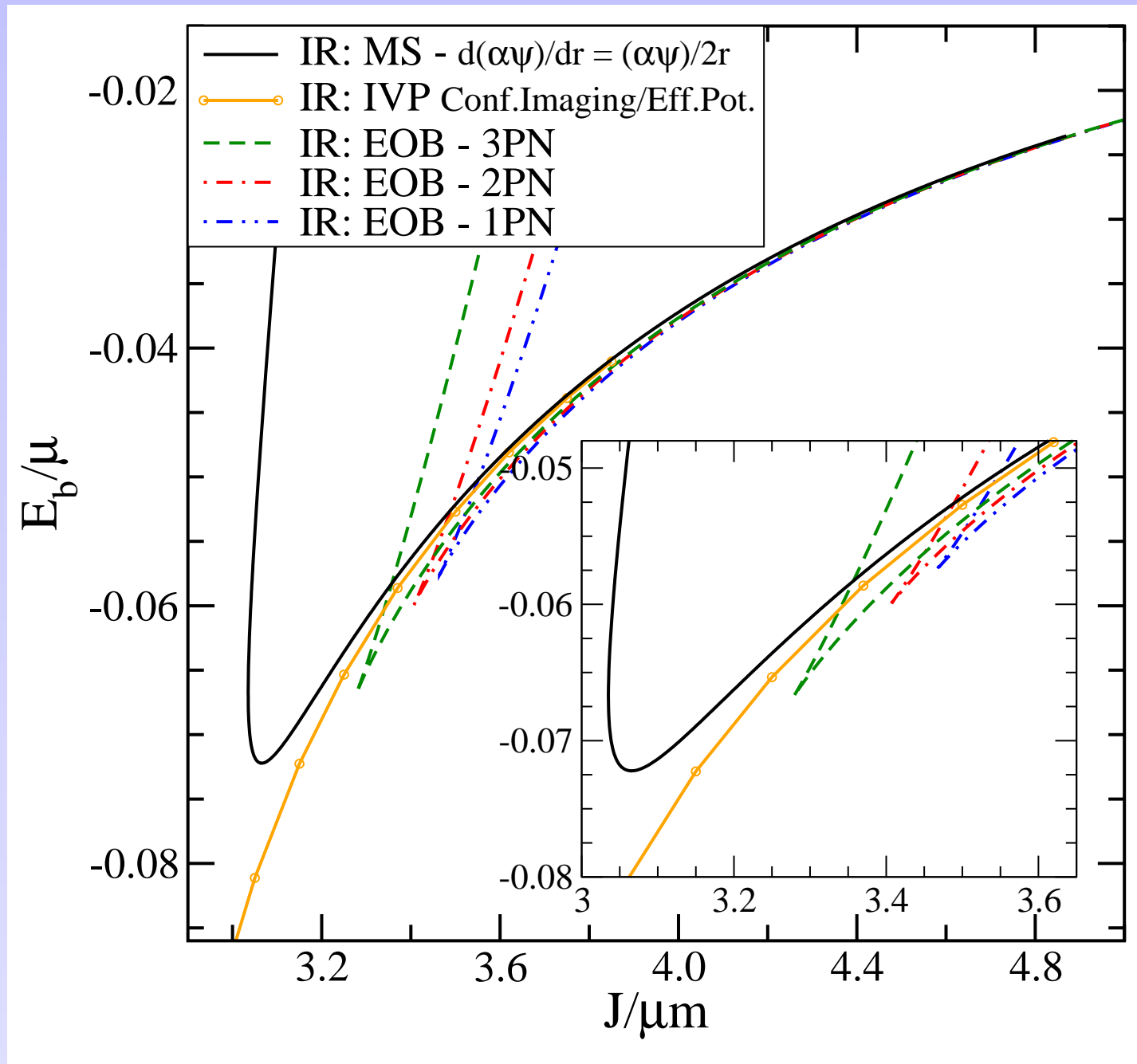
The only remaining freedom in the system is the choice of the lapse boundary condition, the initial spatial and temporal gauge, and the initial dynamical (“wave”) content found in

$\alpha|_S$ ,  $\tilde{\gamma}_{ij}$  and  $K$ .

# Corotating; Maximal Slice; Comparison; $E_b/\mu$ vs $J/\mu m$



# Irrotational; Maximal Slice; Comparison; $E_b/\mu$ vs $J/\mu m$



# Are We Finding Quasicircular orbits?

- The Komar mass criteria is used to choose the quasiequilibrium model which implies quasicircular orbits.
- Can we find an independent way of verifying this?
  - Comparison with post-Newtonian:
    1. Compare plots as we have done already.
    2. Look at post-Newtonian definitions of ellipticity (eg. Mora & Will[14] or Memmesheimer, Gopakumar, & Schäfer[13]).
  - An effective potential approach[6, 8].

# Measuring the Spin of a Black Hole

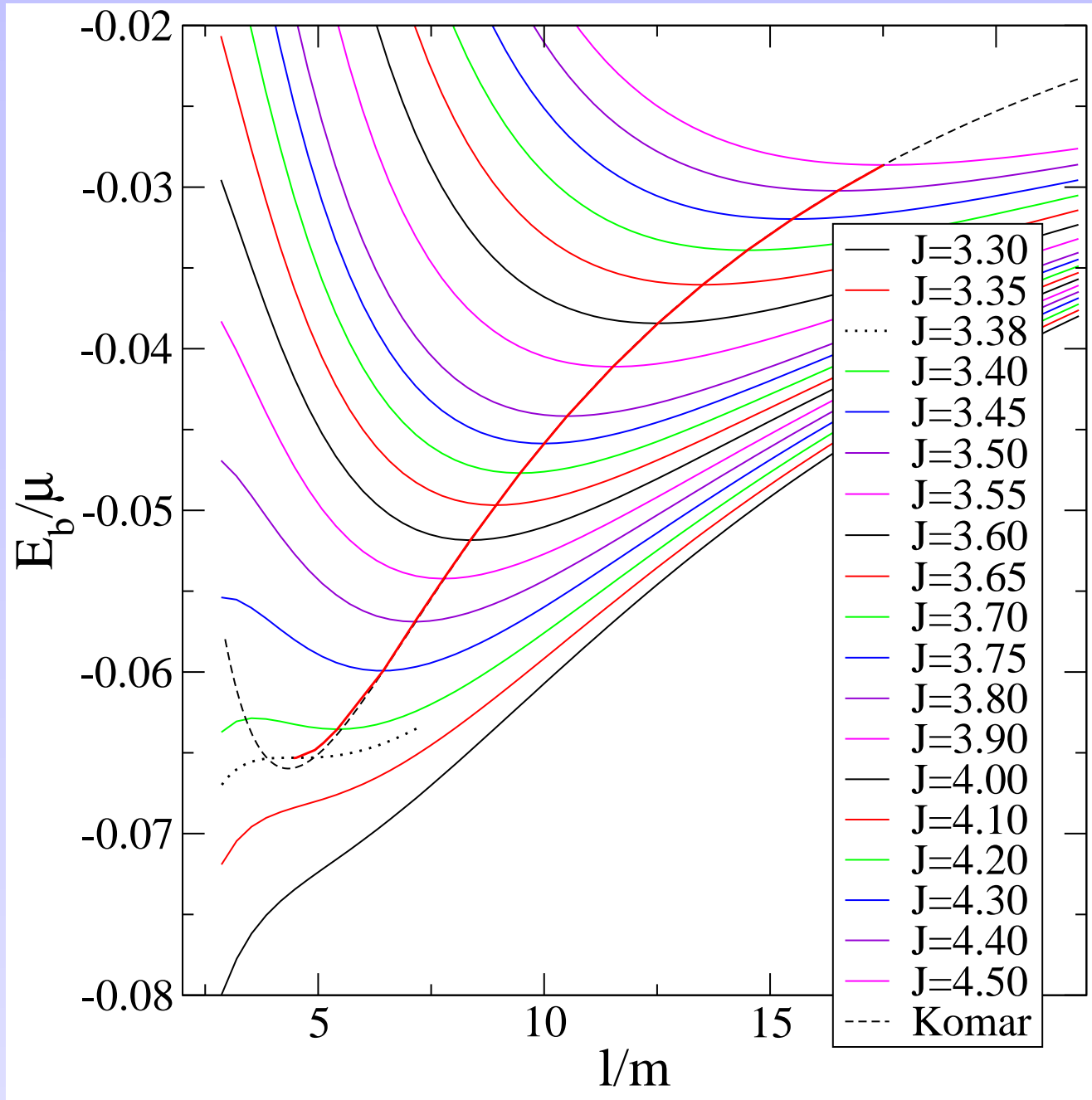
- Spin is only rigorously defined at spatial/null infinity.
- Must use *quasi-local* definition: e.g. Brown & York[5] or Ashtekar & Krishnan[2]

$$\begin{aligned} S &= -\frac{1}{8\pi} \oint_{BH} K_{ij} \xi^i s^j \sqrt{h} d^2x \\ &= -\frac{1}{8\pi} \oint_{BH} \tilde{A}_{ij} \xi^i \tilde{s}^j \sqrt{\tilde{h}} d^2x \end{aligned}$$

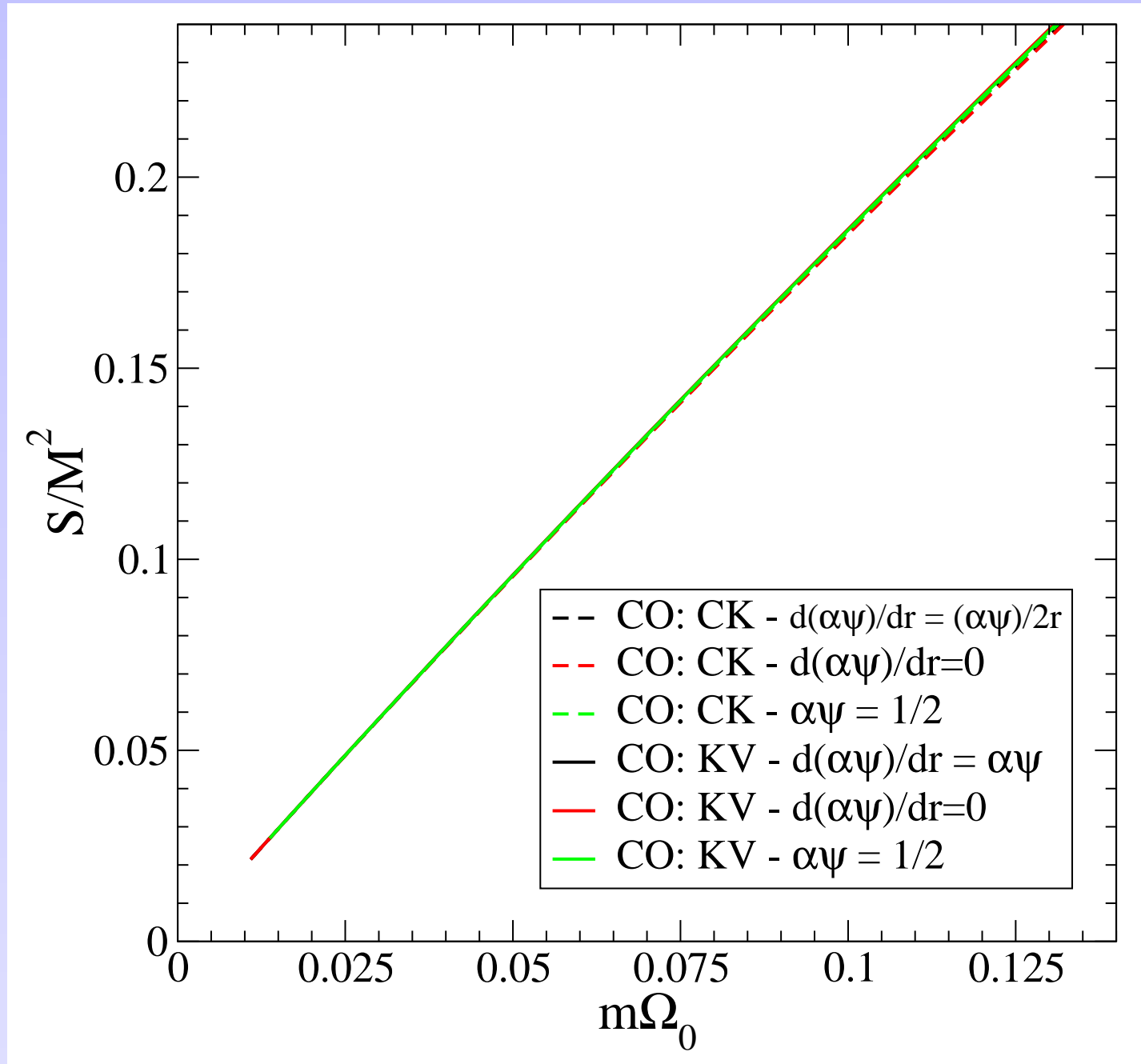
$$\xi^i \tilde{s}_i = 0$$

$$\xi^i = \begin{cases} \xi_{\text{CK}}^i & : \text{ Killing vector of } \tilde{h}_{ij} \Rightarrow \text{conformal Killing vector of } h_{ij} \\ \xi_{\text{KV}}^i & : \text{ Killing vector of } h_{ij} \end{cases}$$

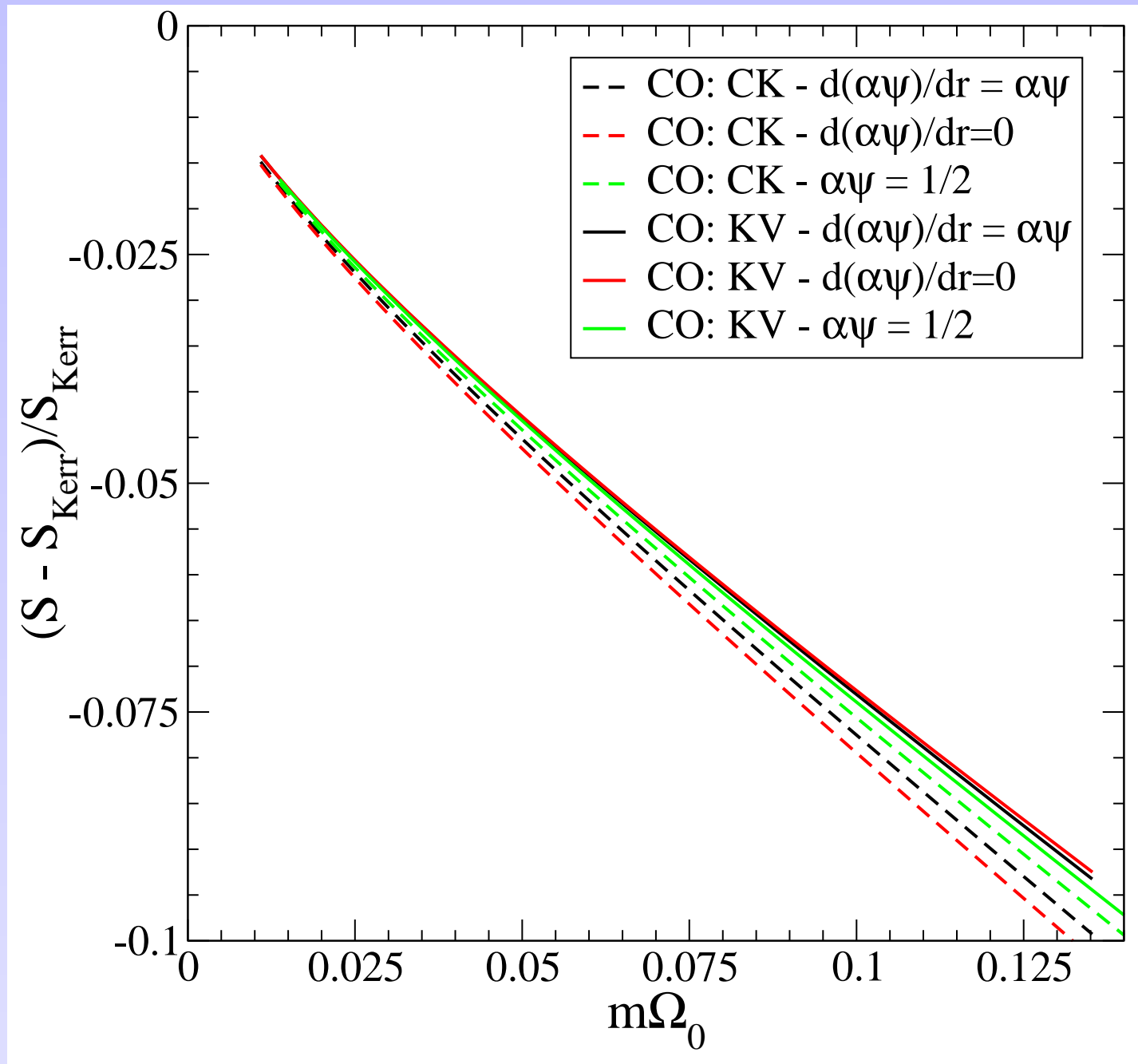
# Corotation: Effective Potential; $E_b/\mu$ vs $\ell/m$



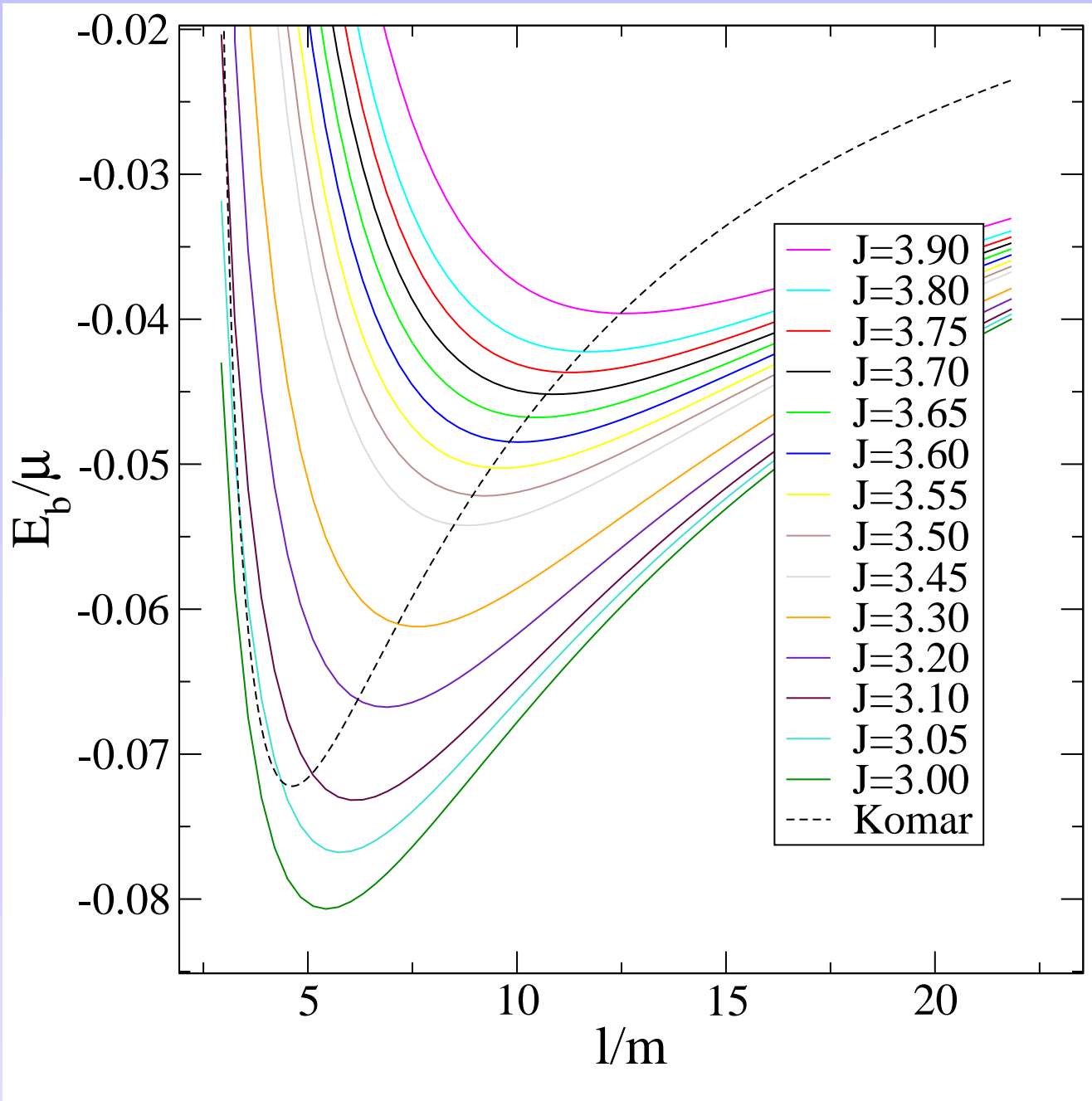
# Corotation: Flat & True KV Spin; $S/M_{irr}^2$ vs $m\Omega_0$



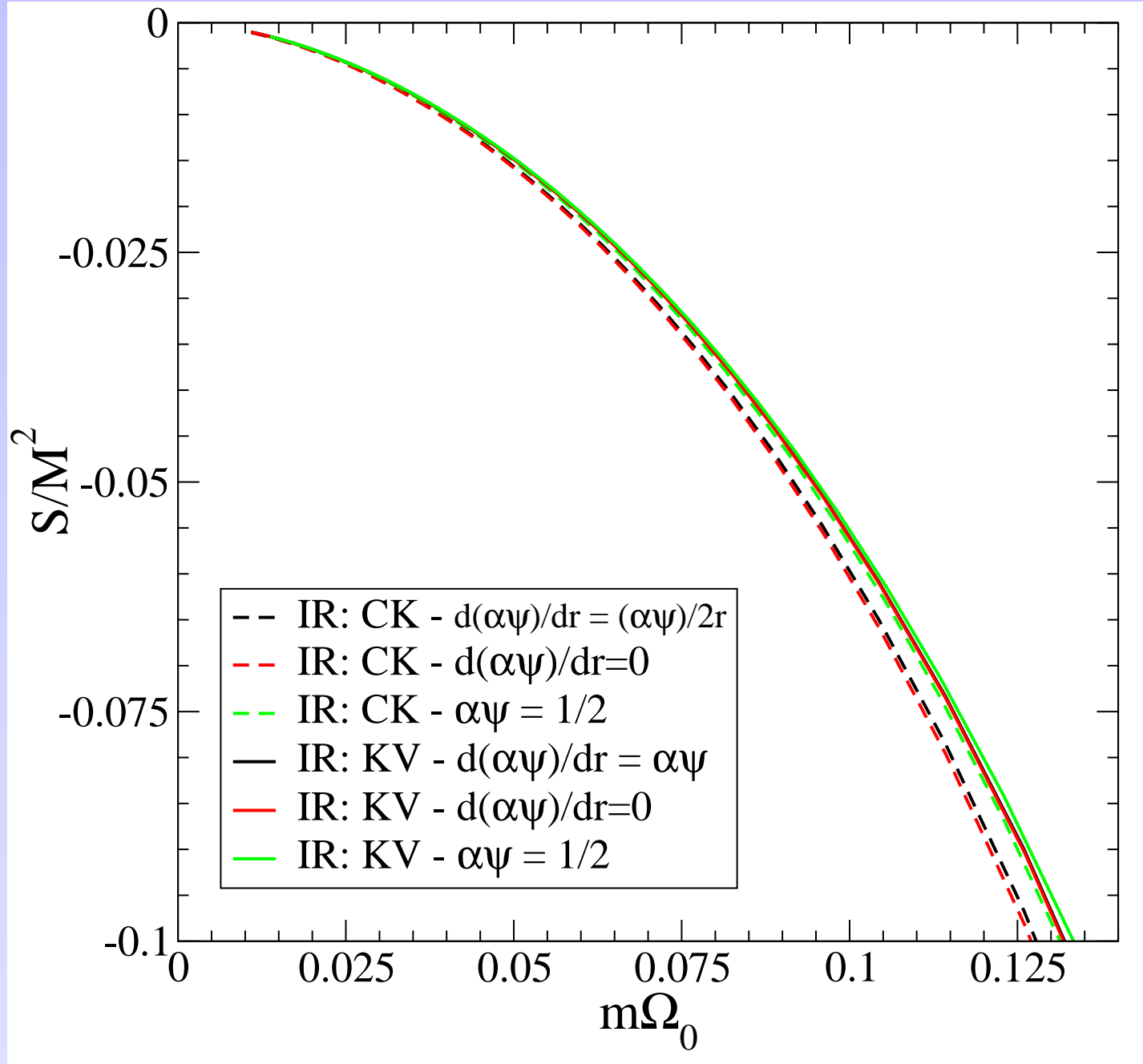
# Corotation: Flat & True KV Spin Error; $(S - S_{Kerr})/S_{Kerr}$ vs $m\Omega_0$



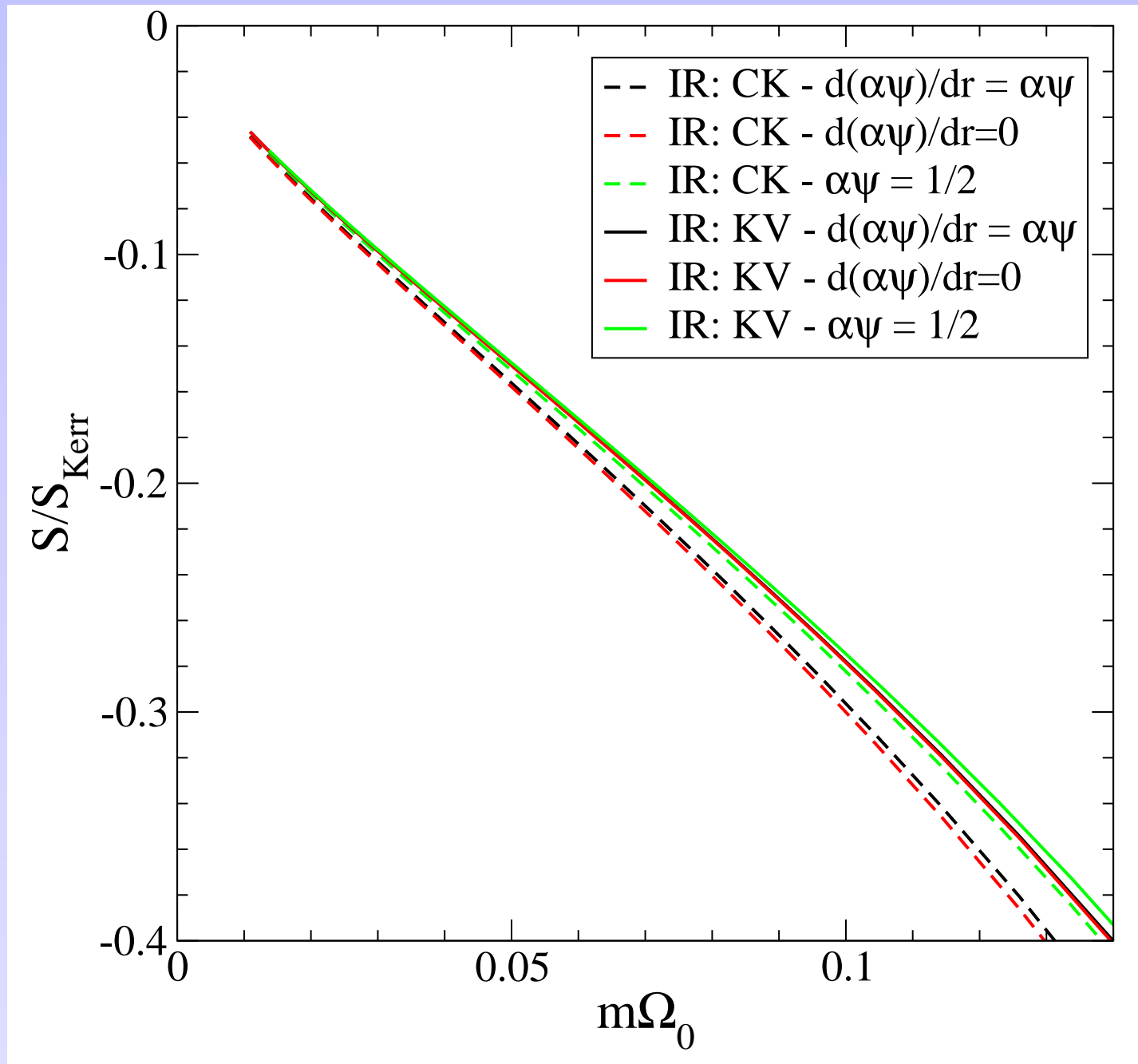
# Irrotation: Effective Potential; $E_b/\mu$ vs $\ell/m$



# Irrotation: Flat & True KV Spin; $S/M_{irr}^2$ vs $m\Omega_0$



# Irrotation: Flat & True KV Spin Error; $S/S_{Kerr}$ vs $m\Omega_0$



# True Irrotation

Clearly the 0<sup>th</sup>-order notion of defining “irrotation” is not adequate

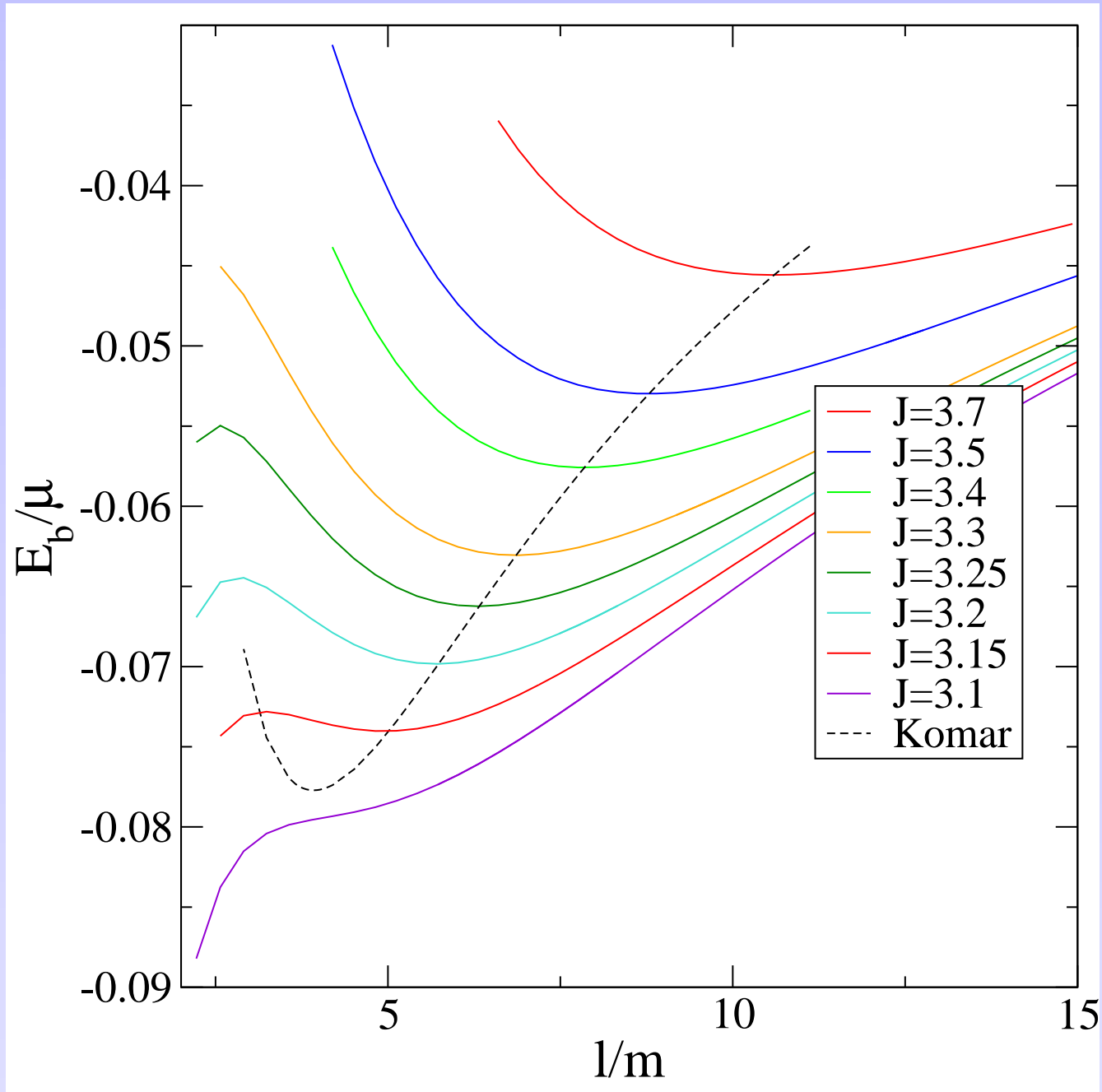
$$\beta^i|_{\mathcal{S}} = \alpha\psi^{-2}\tilde{s}^i|_{\mathcal{S}} + \Omega_0\xi^i|_{\mathcal{S}}.$$

Instead, choose

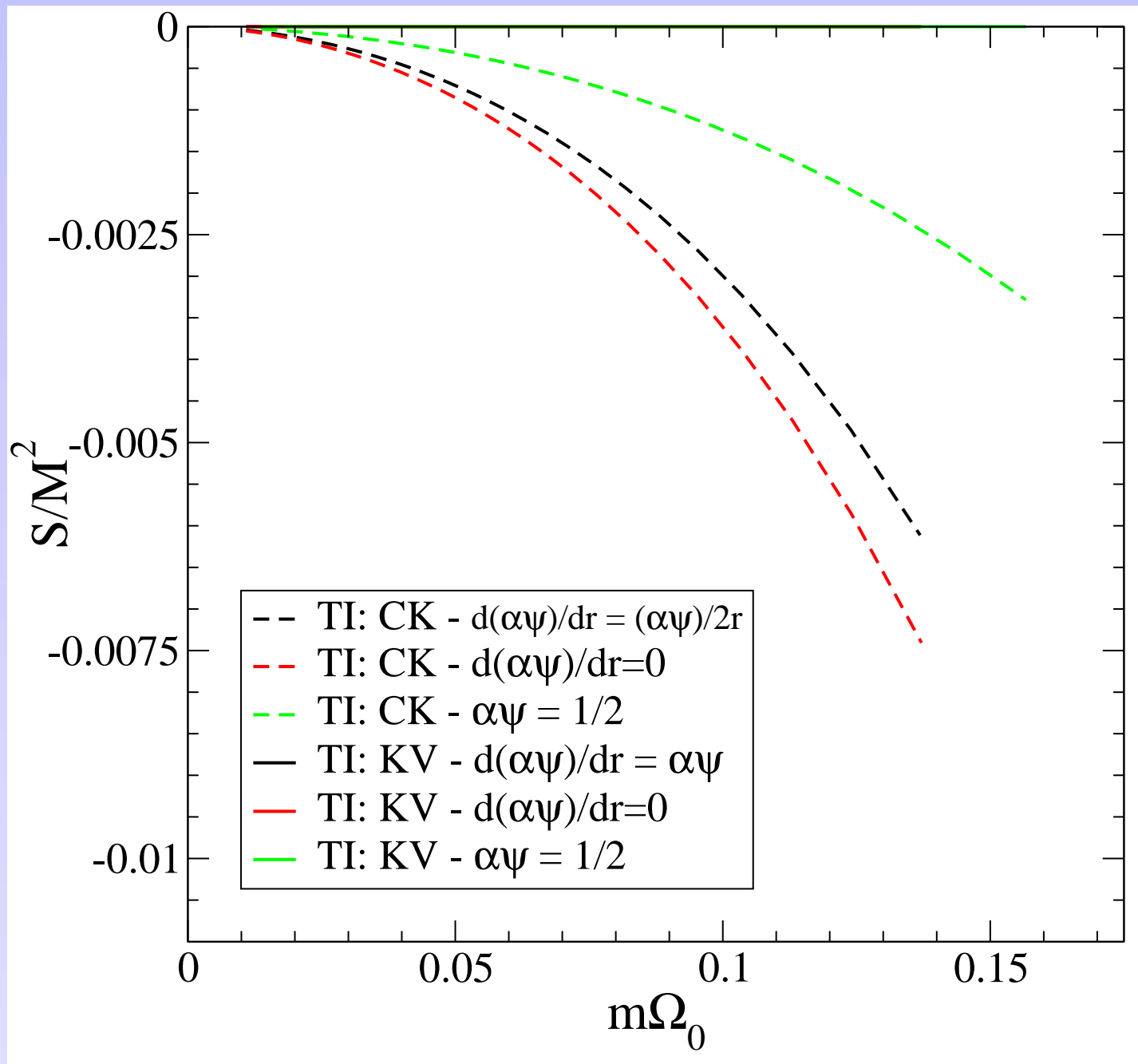
$$\beta^i|_{\mathcal{S}} = \alpha\psi^{-2}\tilde{s}^i|_{\mathcal{S}} + \Omega_{\text{BH}}\xi^i|_{\mathcal{S}}$$

and choose  $\Omega_{\text{BH}}$  so that the quasilocal spin vanishes.

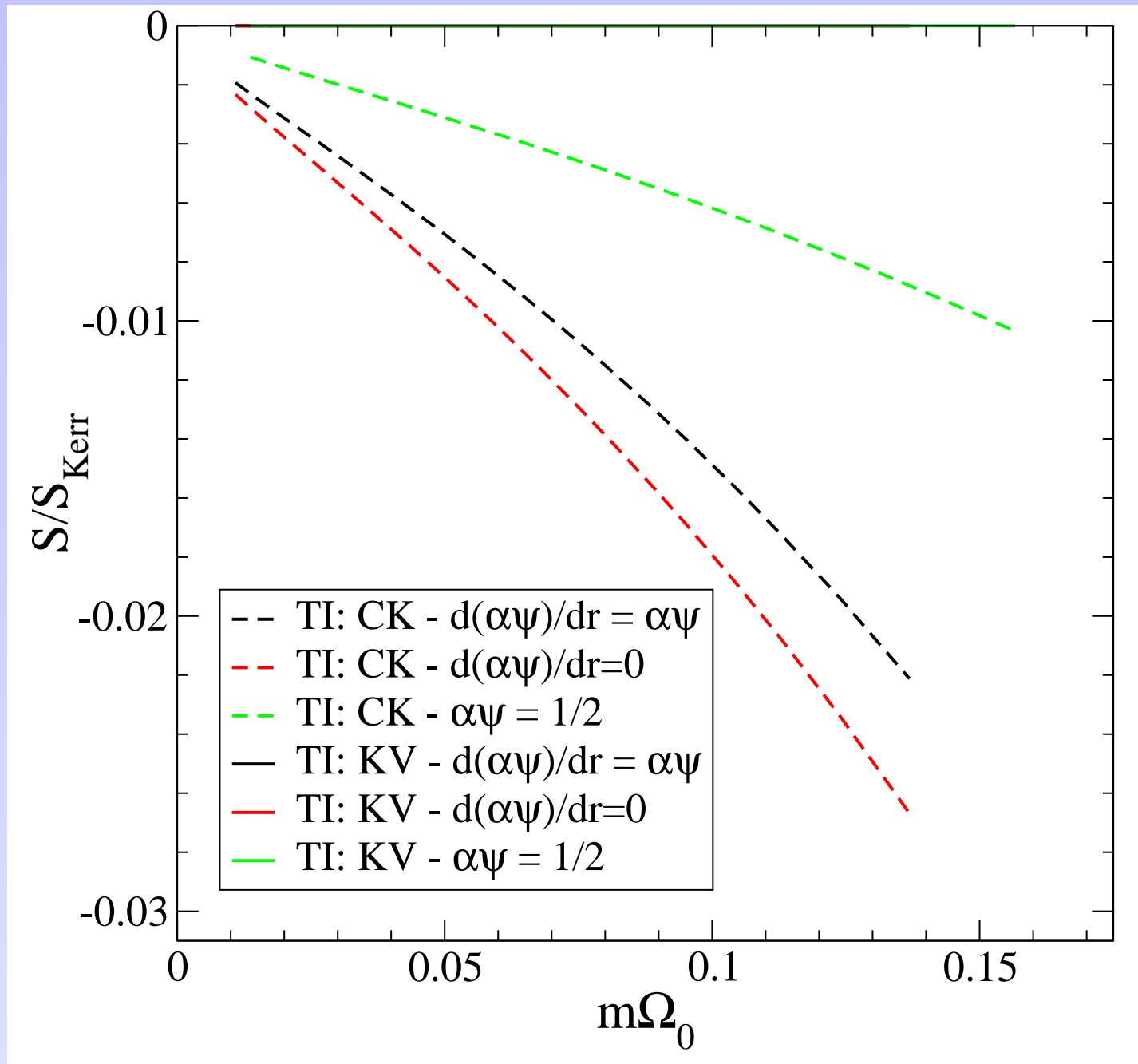
# True Irrotation: Effective Potential; $E_b/\mu$ vs $l/m$



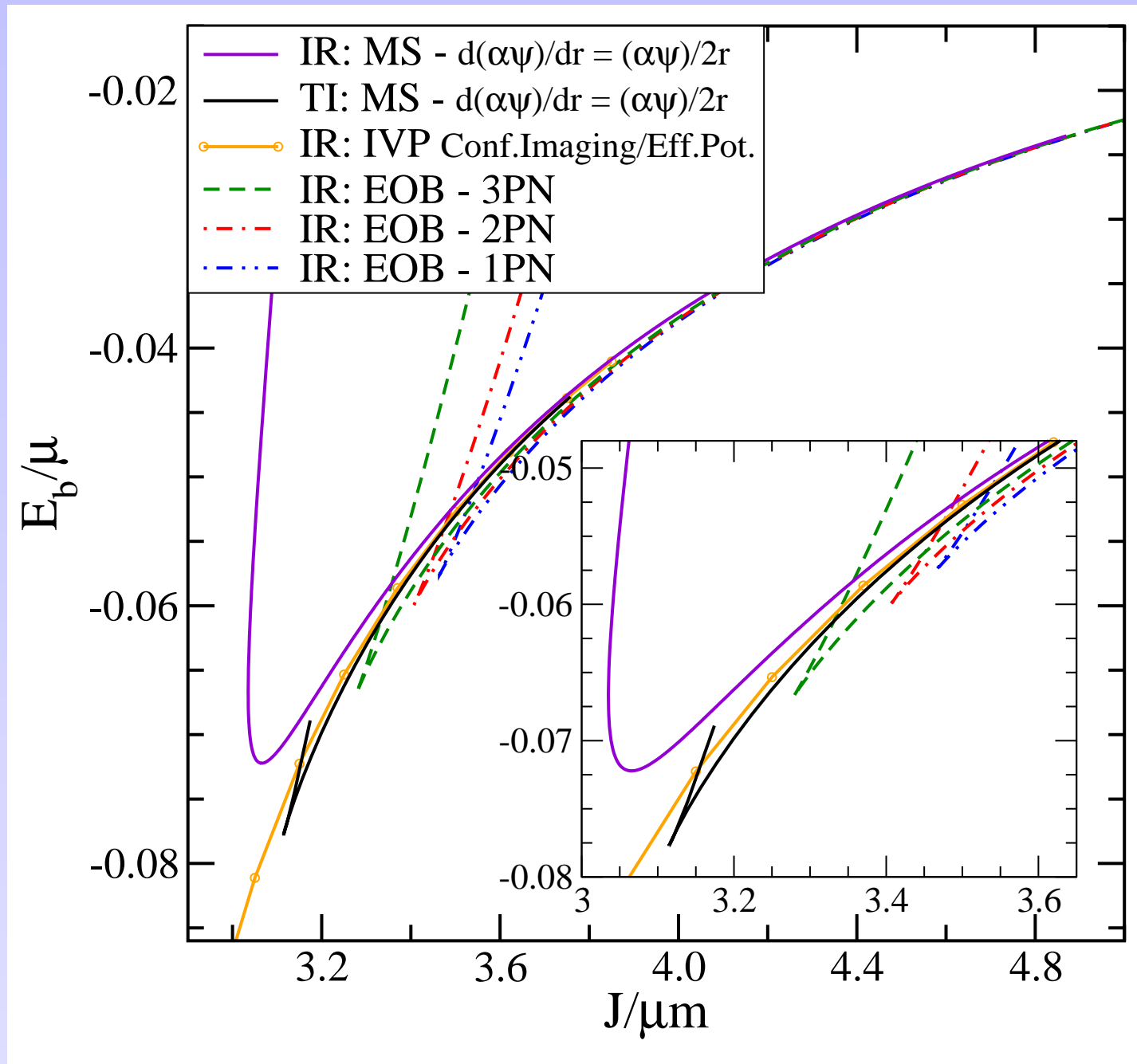
# True Irrotation: Flat & True KV Spin; $S/M_{irr}^2$ vs $m\Omega_0$



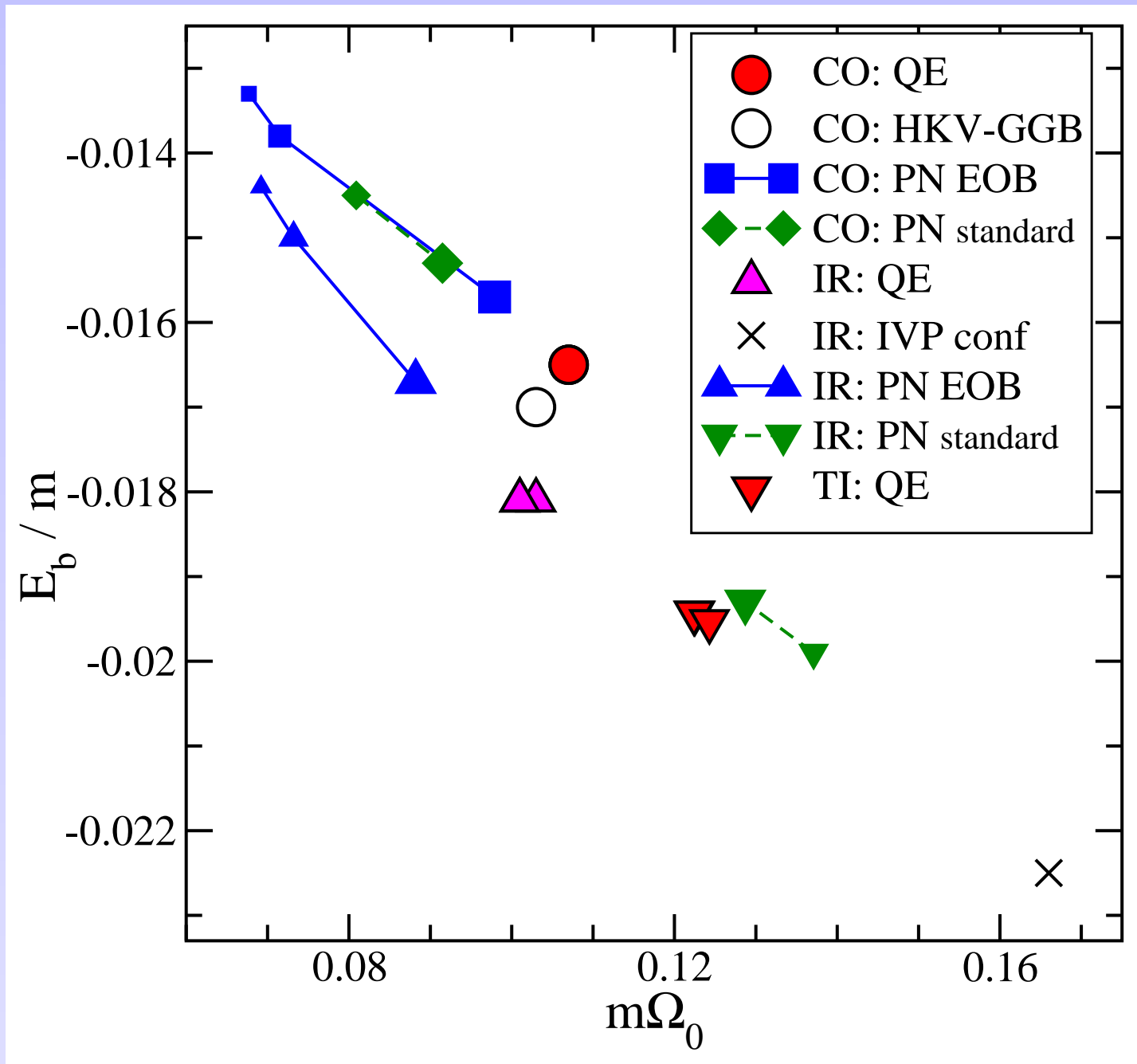
# True Irrotation: Flat & True KV Spin Error; $S/S_{Kerr}$ vs $m\Omega_0$



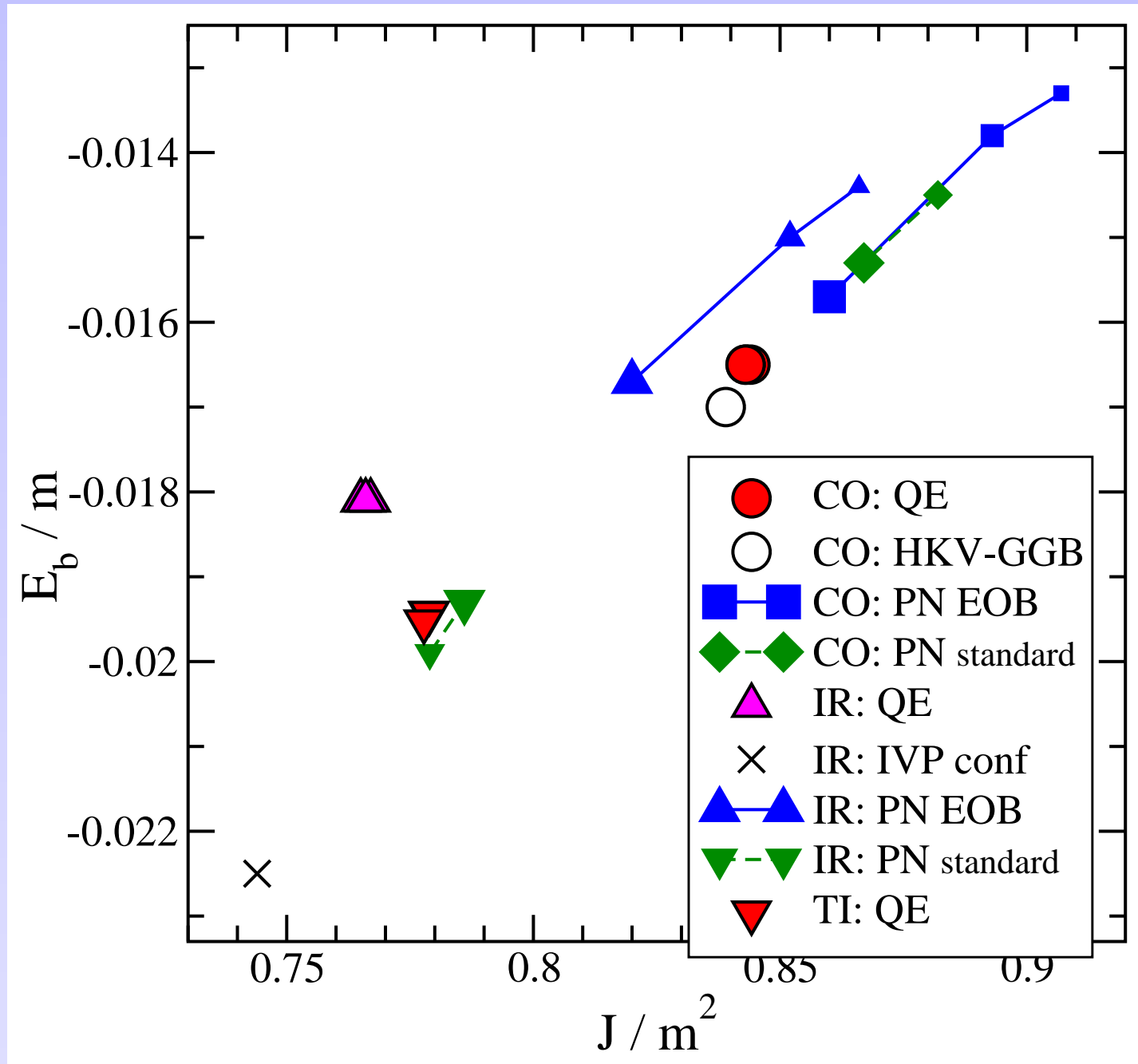
# True Irrotation; Maximal Slice; Comparison; $E_b/\mu$ vs $J/\mu m$



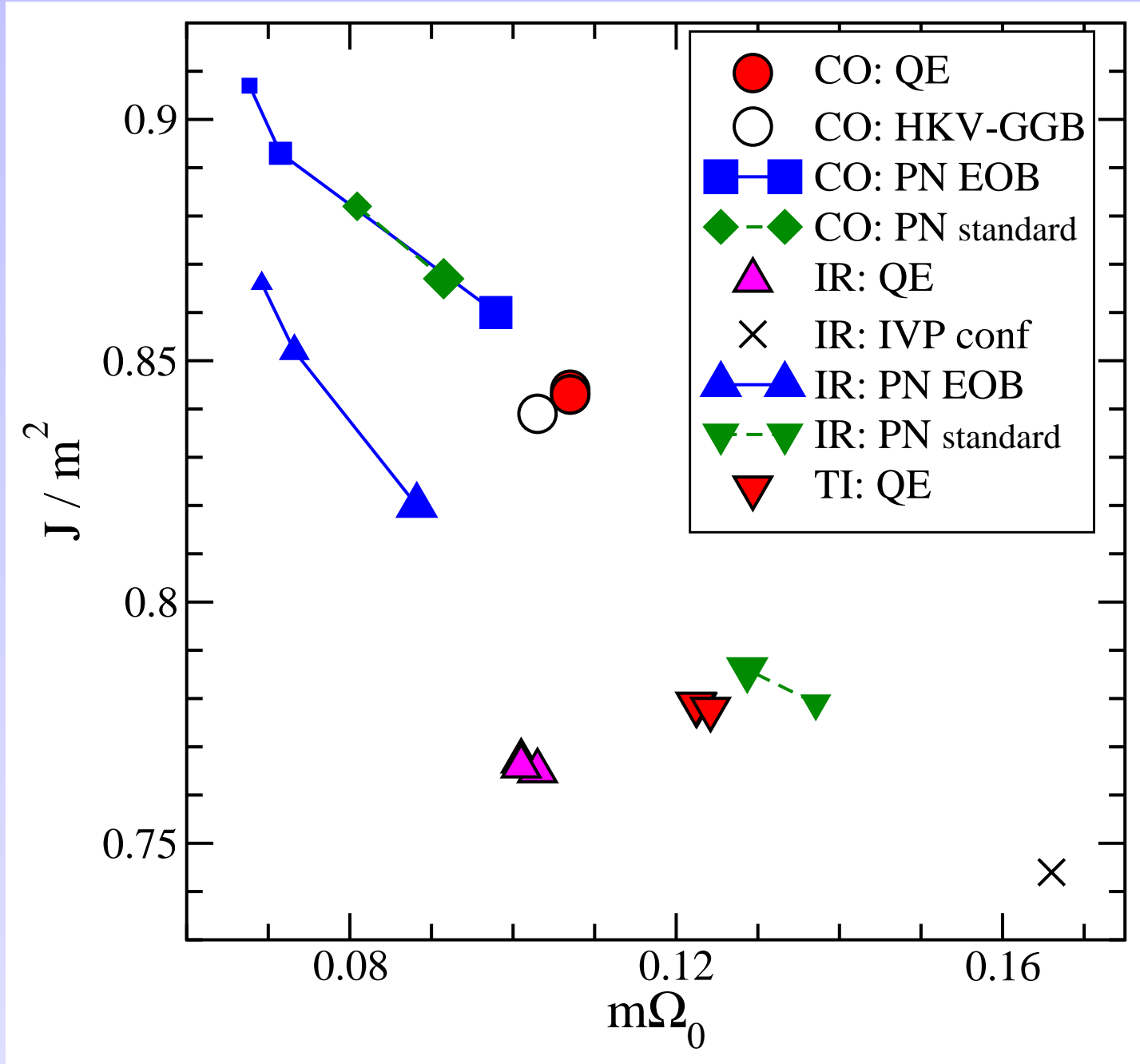
# Maximal Slice; Comparison of ISCO; $E_b/M_{irr}$ vs $\Omega M_{irr}$



# Maximal Slice; Comparison of ISCO; $E_b/M_{irr}$ vs $J/M_{irr}^2$



# Maximal Slice; Comparison of ISCO; $J/M_{irr}^2$ vs $\Omega M_{irr}$



# Understanding the Higher Order Effects

- Irrotation

$\Omega_{\text{BH}}$  is the amount by which we must “counter-spin” the hole to achieve no angular momentum:  $\Omega_{\text{BH}} \leq \Omega_0$

- Corotation

We find that  $\frac{S}{M^2}|_{\text{QL}} \leq \frac{S}{M^2}|_{\text{Kerr}(\Omega_0)}$ . In this context, if we let  $\Omega_{\text{BH}}$  represent the angular velocity of a Kerr BH with spin  $S|_{\text{QL}}$ :  $\Omega_{\text{BH}} \leq \Omega_0$

$$\frac{\Omega_{\text{BH}}}{\Omega_0} = \frac{1}{m\Omega_0} \frac{\frac{S}{M^2}|_{\text{QL}}}{\sqrt{4 + \left(\frac{S}{M^2}|_{\text{QL}}\right)^2}}$$

# Tidal Field Rotation Rate

From the local inertial frame of one black hole, the tidal field of the second black hole is rotating with an angular velocity  $\omega$ [1]

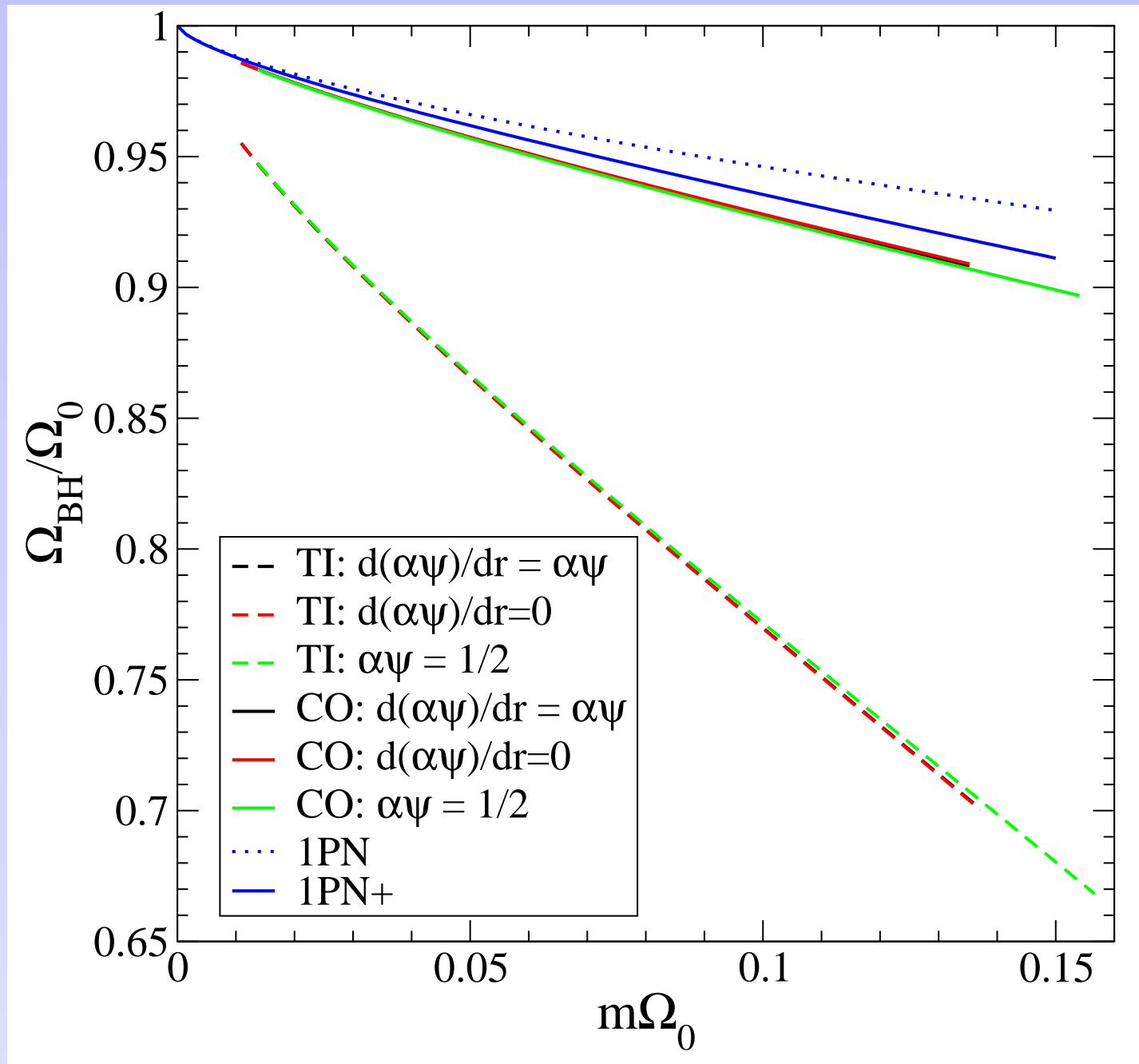
$$\omega = \Omega_0 \left[ 1 - \eta \left( \frac{m}{b} \right) + O \left( \frac{m}{b} \right)^{3/2} \right]$$

To 1PN order

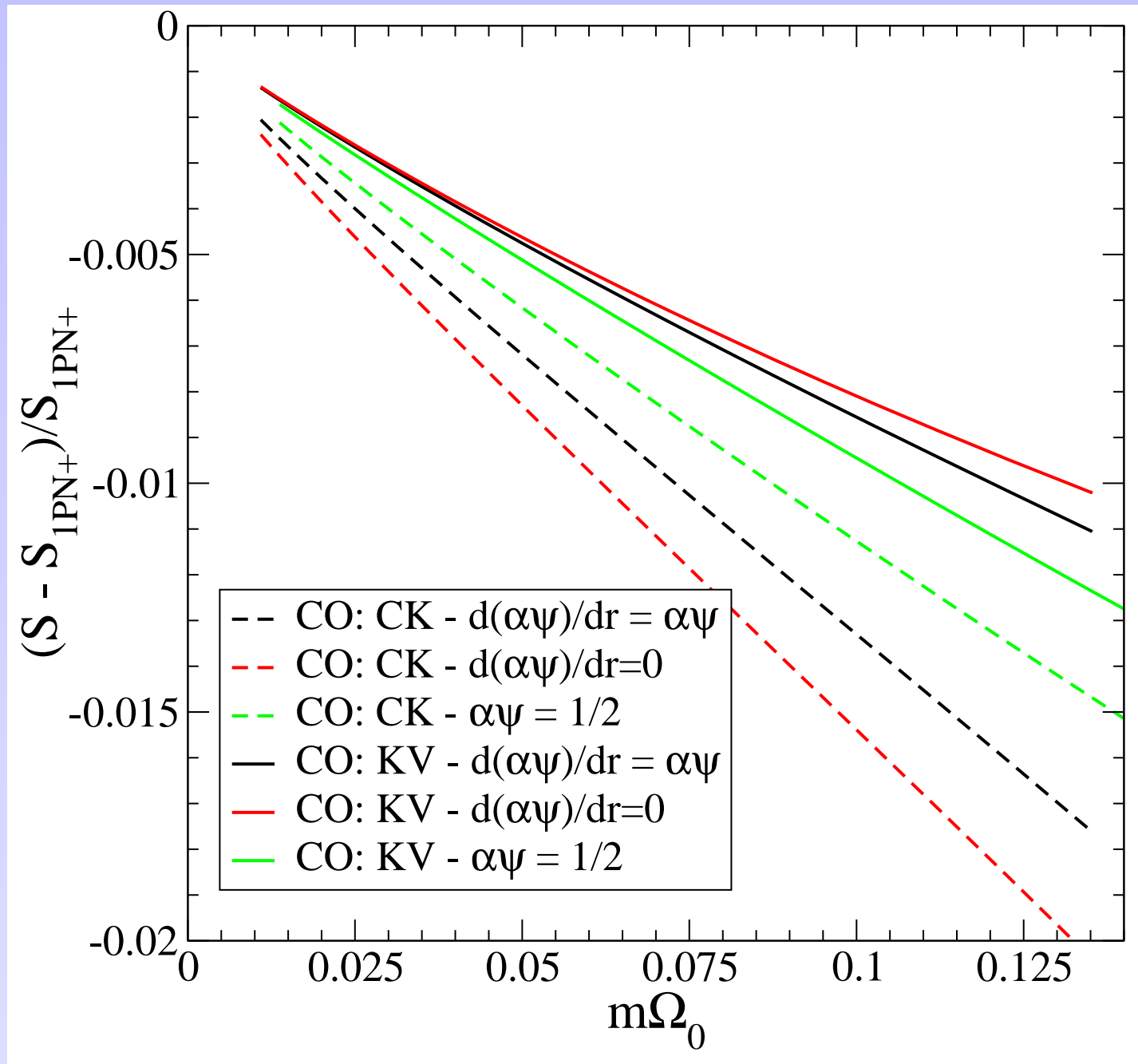
$$\frac{m}{b} = (m\Omega_0)^{2/3} \left[ 1 + \left( 1 - \frac{1}{3}\eta \right) (m\Omega_0)^{2/3} + O(m\Omega_0)^{4/3} \right]$$

$$\frac{\Omega_{\text{BH}}}{\Omega_0} = \frac{\omega}{\Omega_0} = 1 - \eta (m\Omega_0)^{2/3} - \eta \left( 1 - \frac{1}{3}\eta \right) (m\Omega_0)^{4/3} + \dots$$

# Ratio of Spin and Orbital Angular Velocities; $\Omega_{\text{BH}}/\Omega_0$ vs $m\Omega_0$



# Spin Error Using 1PN Kerr Spin; $(S - S_{1PN+})/S_{1PN+}$ vs $m\Omega_0$



## References

- [1] K. Alvi. Approximate binary-black-hole metric. *Phys. Rev. D*, 61:124013/1–19, June 2000. 31
- [2] A. Ashtekar and B. Krishnan. Dynamical horizons and their properties. *Phys. Rev. D*, 68:104030/1–25, 2003. 15
- [3] T. W. Baumgarte, G. B. Cook, M. A. Scheel, S. L. Shapiro, and S. A. Teukolsky. General relativistic models of binary neutron stars in quasiequilibrium. *Phys. Rev. D*, 57:7299–7311, June 1998. 4
- [4] S. Bonazzola, E. Gourgoulhon, and J.-A. Marck. Numerical models of irrotational binary neutron stars in general relativity. *Phys. Rev. Lett.*, 82:892–895, Feb. 1999. 4
- [5] J. D. Brown and J. W. York, Jr. Quasilocal energy and conserved charges derived from the gravitational action. *Phys. Rev. D*, 47:1407–1419, 1993. 15
- [6] G. B. Cook. Three-dimensional initial data for the collision of two black holes II: Quasicircular orbits for equal mass black holes. *Phys. Rev. D*, 50:5025–5032, Oct. 1994. 14
- [7] G. B. Cook and H. P. Pfeiffer. Excision boundary conditions for black hole initial data. *Phys. Rev. D*, 70:104016/1–24, 2004. 0
- [8] S. Detweiler. Periodic solutions of the Einstein equations for binary systems. *Phys. Rev. D*, 50:4929–4943, Oct. 1994. 14
- [9] D. M. Eardley. Black hole boundary conditions and coordinate conditions. *Phys. Rev. D*, 57:2299–2304, Feb. 1998. 9
- [10] E. Gourgoulhon, P. Grandclément, and S. Bonazzola. Binary black holes in circular orbits. I. A global spacetime approach. *Phys. Rev. D*, 65:044020/1–19, Feb. 2002. 4, 10
- [11] P. Grandclément, E. Gourgoulhon, and S. Bonazzola. Binary black holes in circular orbits. II. Numerical methods and first results. *Phys. Rev. D*, 65:044021/1–18, Feb. 2002. 4, 10

- [12] P. Marronetti, G. J. Mathews, and J. R. Wilson. Binary neutron-star systems: From the Newtonian regime to the last stable orbit. *Phys. Rev. D*, 58:107503/1–4, Nov. 1998. 4
- [13] R.-M. Memmesheimer, A. Gopakumar, and G. Schäfer. Third post-newtonian accurate generalized quasi-keplerian parametrization for compact binaries in eccentric orbits. *Phys. Rev. D*, 70:104011, Nov. 2004. 14
- [14] T. Mora and C. M. Will. Post-Newtonian diagnostic of quasiequilibrium binary configurations of compact objects. *Phys. Rev. D*, 69:104021/1–25, 2004. 14
- [15] K. Uryū and Y. Eriguchi. New numerical method for constructing quasiequilibrium sequences of irrotational binary neutron stars in general relativity. *Phys. Rev. D*, 61:124023/1–19, June 2000. 4