Noise Kernel near the horizon of de Sitter space

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Abstract.

The symmetric two point correlation function of the stress-energy tensor - also known as the noise kernel - for the conformally invariant scalar field is computed in the static region of de Sitter space in terms of the usual static coordinates. Its behavior near the cosmological horizon is investigated. The exact expression is then compared to an approximate expression derived using a quasi-local expansion in order to test the validity of that expansion in a static spherically symmetric spacetime containing a horizon. This gives insight into the likely range of validity of the quasi-local expansion for the noise kernel in Schwarzschild spacetime which has been previously computed.

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1. Introduction

While studies of quantum field theory in curved space [1] have yielded fundamental results such as black hole evaporation [2], and semiclassical gravity provided the framework for inflationary cosmology [3] and understanding the origin of structures in the early universe [4], the necessity of including the effects of fluctuations and correlations of the stress-energy tensor for quantized fields is becoming increasingly noteworthy. These aspects enter in an essential way in establishing the criteria for the validity of semiclassical gravity [5, 6]. They play an important role in cosmology, due to anticipated new data from planned precision cosmological observation experiments, and in the study of quantum effects in black hole spacetimes. Investigations have been made of their effect on density perturbations during inflation [7, 8, 9, 10, 11]. The backreaction of fluctuations [12] of quantum matter fields may shed some light on the evolution and end-state evaporation of black holes.

One theory which consistently captures the matter field fluctuations and the induced metric fluctuations is stochastic gravity [13, 14]. While semiclassical gravity is based on the semiclassical Einstein equation which has as its source the vacuum expectation values of the stress energy tensor of quantum fields, stochastic gravity has at its heart the Einstein-Langevin equation from which one can study the features and dynamics of induced‡ metric fluctuations (sometimes called 'spacetime foam'). This equation is driven by the symmetrized two-point function of the stress-energy tensor of the quantum fields, called the noise kernel.

For the past decade, much of the focus of stochastic gravity has been on the calculation of the noise kernel (and more generally the two-point function for the stress-energy tensor) for free fields in various spacetimes. Beginning with the derivation of a general expression for the noise kernel in arbitrary spacetimes in terms of products of derivatives of the Wightman function [15, 16, 17], computations of the noise kernel have been made in Minkowski [12, 18, 19, 20, 21], de Sitter [16, 22, 23], anti-de Sitter [23, 24], Schwarzschild [20, 25], and Tolman-Bondi [26] spacetimes.

One particular area of interest in stochastic gravity is the behavior of the noise kernel near black holes. In general the black hole backreaction and fluctuations problem [27] entails studying the effects upon the evolution of a black hole of both the emitted Hawking radiation and the metric fluctuations driven by the fluctuations of the quantum field. Sinha, Raval and Hu [28] outlined a program for such an investigation involving the stochastic gravity upgrade (via the Einstein-Langevin equation) of investigations carried out for the mean field in semiclassical gravity (through the semiclassical Einstein equation) by York [29, 30] and by York and his collaborators [31]. However, what stalls the progress in this program is that the radial function of the Schwarzschild metric is only accessible numerically (not to mention the solution to the semiclassical Einstein equation, which is the correct background solution to use in the

‡ Induced as opposed to intrinsic fluctuations due to the linear response of the semiclassical equations to metric perturbations; see [11] for a discussion thereof.

solution of the Einstein-Langevin equation). This leads us to wonder if there is an alternative way to investigate the behavior of quantum fluctuations near a horizon.

de Sitter space, represented in the usual static coordinates, offers such a way. Because static de Sitter space has a metric structure that is mathematically of the same form as that of Schwarzschild spacetime, it is expected that the noise kernel, when evaluated in the these coordinates in the region near the cosmological horizon, will have the same type of behavior as the noise kernel near the event horizon in Schwarzschild spacetime.

Previously Roura and Verdaguer [16] computed the noise kernel of a massless, minimally coupled scalar field in the Bunch-Davies vacuum state for spacelike separations of points. Pérez-Nadal, Roura, and Verdaguer [22] extended this calculation to massive and massless fields in arbitrary dimensions. Osborn and Shore [23] computed the two-point function for the stress-energy tensor for conformally invariant free fields in spaces with constant curvature which include the Euclidean sector of de Sitter space. Ford and collaborators [10] computed the two-point correlation function for the energy density of the conformally invariant scalar field in de Sitter space as part of an investigation of density perturbations arising from fluctuations of quantum fields during inflation.

To carry out the comparison between the exact noise kernel in de Sitter space in the usual static coordinates and expansions of it in terms of those coordinates, it is necessary to obtain an exact and explicit expression for the noise kernel. There are several ways that such an expression could be obtained. For example the expression for the stress-energy tensor two-point function derived by Osborn and Shore [23] for Euclidean de Sitter space could be analytically continued to the Lorentzian sector. Alternatively one could take the expression by Campos and Verdaguer [32] for the influence action in a spatially flat RW spacetime, compute the second variation with respect to the metric perturbations, evaluate using the scale factor for de Sitter space in the spatially flat slicing, and finally compute the Fourier integrals to yeild an explicit exression in terms of spatial coordinates. Another possibility is to compute the noise kernel of the Minkowski vacuum in Rindler space, where it is a thermal state, and conformally transform directly to de Sitter in the usual static coordinates making use of the conformal correspondence between Rindler and static de Sitter.

We have chosen yet another alternative which is to directly compute the noise kernel in flat space in the usual Cartesian coordinates and conformally transform it to de Sitter space in the usual spatially flat coordinates. Although a previous calculation of the noise kernel in Minkowksi space for the massless scalar field in the Minkowski vacuum was carried out by Martin and Vergaguer [18, 19], these calculations were in terms of Fourier components. No explicit, closed form expression for all components in terms of coordinates has been presented for an arbitrary separation of points. We present such an expression here.

Using this expression, we compute an exact, closed form expression for the noise kernel for the conformally invariant field in the Bunch-Davies state in de Sitter space in the comoving coordinates by means of a conformal transformation, which we then evaluate in the usual static coordinates by means of a coordinate transformation§; this ensures that the state in the static coordinates is the Gibbons-Hawking state [33]. This expression is used to investigate the behavior of the noise kernel inside and on the cosmological horizon. In particular, we find that the noise kernel (when expressed in an orthonormal frame) remains finite as either point approaches the cosmological horizon so long as the points are non-null separated.

It is expected that, in the near horizon region, this behavior will be similar to that of the noise kernel for a non-rotating black hole when the field is in the Hartle-Hawking state [34]. Thus, we are interested in comparing this expression with that derived in [20] for Schwarzschild spacetime. In that work, an approximate expression for the noise kernel for the conformally invariant scalar field was obtained by means of a quasi-local expansion. We produce an equivalent expression in de Sitter space by writing the exact noise kernel as a series expansion in terms of the coordinate separation, and then truncating this expansion at the order used in [20]. We consider the cases in which the separation is in the time direction and when it is in the radial direction. In both cases the result is an expansion which depends on $|g_{tt}|$ for the metric (18). For the time separation, to leading order in powers of $|g_{tt}|^{-1}$ the expansion is exactly the same as that of Ref. [20] except that of course the specific form of $|g_{tt}|$ is different for the two cases (thus, in the near horizon limit the behavior of the expansions is identical). For radial separations the two expansions also differ by a scaling of the radial coordinate.

We expect this type of expansion to be valid for separations of points on the order of the mass scale in Schwarzschild spacetime. However, it is not known if the quasi-local approximation scheme remains valid when either of the points approaches the horizon. To test this, we compute the relative error introduced by truncating the expansion in de Sitter space and then comparing with the exact result in de Sitter space. For a separation in the time direction we find that so long as the two points are not on the horizon the relative error depends only on the separation in time between the two points. For a separation in the radial direction we find that the expansion breaks down completely if either point is on the horizon. If both points are away from the horizon then the relative error depends both on the separation between the two points and on the distance from the horizon to the closest point to the horizon, becoming larger as this distance gets smaller for a fixed point separation.

This paper is organized as follows: in Sec. 2, we present a closed form expression for all components of the noise kernel for the conformally invariant scalar field in the conformal vacuum state of a large class of conformally flat spacetimes. This expression is obtained by means of a conformal transformation from Minkowski spacetime using the conformal transformation law derived in [20]. It is valid for arbitrary separations of points, but includes formally divergent quantities when the points are null separated; however, these quantities cancel for integrals over the noise kernel in such a way that

§ The results of the three alternative routes presented above may be recovered using our method; we show this explicitly for the Osborn and Shore result in the appendix.

the final expression is a well-defined distribution on spacetime. In Sec. 3, we show the corresponding expression for two components of the noise kernel in the static de Sitter coordinates when the points are separated in a timelike or spacelike direction. The behavior of the noise kernel as one or both of the points approaches the cosmological horizon is investigated. We also present an expansion for one component of the noise kernel in inverse powers of the coordinate separation for separations in the time direction and separations in the radial direction. The results are compared with the corresponding exact expressions for this component. Our results are summarized and discussed in Sec. 4. Finally, in the appendix, we compare the expression for the noise kernel obtained in Sec. 2 with prior expressions computed in the Euclidean sector of de Sitter space by Osborn and Shore [23] and the Euclidean sector of anti-de Sitter space by Cho and Hu [24] for the two-point function of the stress-energy tensor for the conformally invariant scalar field.

The conventions used throughout are those of Misner, Thorne, and Wheeler [35]. Units are chosen such that $\hbar = c = G = 1$.

2. The noise kernel in conformally flat spacetimes

We compute the noise kernel for the conformally invariant field in de Sitter space in the static coordinates by obtaining a closed form expression for the noise kernel for this field in Minkowski space and make use of its conformal transformation properties demonstrated in [20]. This provides us with a convenient means to compute exact, explicit closed form expressions for the noise kernel in de Sitter space as well as a large class of conformally flat spacetimes - those with metrics that are conformal to the full Minkowski space (or at least enough of it to contain a Cauchy surface).‖

The noise kernel for a quantum scalar field in a Gaussian state can be expressed in terms of the Wightman function [17, 20]. For the conformally invariant scalar field, this expression is

$$N_{abc'd'} = \mathrm{Re}\left\{\bar{K}_{abc'd'} + g_{ab}\bar{K}_{c'd'} + g_{c'd'}\bar{K}'_{ab} + g_{ab}g_{c'd'}\bar{K}\right\} \tag{1}$$

with¶

$$\begin{aligned} 9\bar{K}_{abc'd'} = {} & 4\,\left(G_{;c'b}\,G_{;d'a} + G_{;c'a}\,G_{;d'b}\right) + G_{;c'd'}\,G_{;ab} + G\,G_{;abc'd'} \\ & - 2\,\left(G_{;b}\,G_{;c'ad'} + G_{;a}\,G_{;c'bd'} + G_{;d'}\,G_{;abc'} + G_{;c'}\,G_{;abd'}\right) \\ & + 2\,\left(G_{;a}\,G_{;b}\,R_{c'd'} + G_{;c'}\,G_{;d'}\,R_{ab}\right) \end{aligned}$$

‖ For other conformally flat metrics a different vacuum state is appropriate. For example, de Sitter space in either the comoving or the closed coordinates is conformal to Minkowski space (with $t > 0$ for the comoving coordinates), and the conformal vacuum is the Minkowski vacuum specified by some constant time Cauchy surface. Alternatively, in the usual static coordinates de Sitter space is conformal to Rindler space, and it is therefore the Rindler vacuum which is the preferred conformal vacuum state [1]. Anti-de Sitter space, in contrast, is conformal to Minkowski space with a timelike boundary (for instance, at $x = 0$). Since all Cauchy surfaces cross the boundary the choice of conformal vacuum must respect the appropriate boundary conditions [37].

¶ Note that the superscript $+$ on G^+ has been omitted for notational simplicity.

$$- \left(G_{;ab}\, R_{c'd'} + G_{;c'd'}\, R_{ab}\right) G + \frac{1}{2} R_{c'd'}\, R_{ab} G^2 \tag{2a}$$

$$\begin{aligned}
36\bar{K}'_{ab} = {}& 8 \left(-G_{;p'b}\, G_{;}{}^{p'}{}_{a} + G_{;b}\, G_{;p'a}{}^{p'} + G_{;a}\, G_{;p'b}{}^{p'}\right) \\
& 4 \left(G_{;}{}^{p'}\, G_{;abp'} - G_{;p'}{}^{p'}\, G_{;ab} - G\, G_{;abp'}{}^{p'}\right) \\
& - 2\, R' \, \left(2\, G_{;a}\, G_{;b} - G\, G_{;ab}\right) \\
& - 2 \, \left(G_{;p'}\, G_{;}{}^{p'} - 2\, G\, G_{;p'}{}^{p'}\right) \, R_{ab} - R'\, R_{ab} G^2
\end{aligned} \tag{2b}$$

$$\begin{aligned}
36\bar{K} = {}& 2\, G_{;p'q}\, G_{;}{}^{p'q} + 4\, \left(G_{;p'}{}^{p'}\, G_{;q}{}^{q} + G\, G_{;p}{}^{p}{}_{q'}{}^{q'}\right) \\
& - 4\, \left(G_{;p}\, G_{;q'}{}^{pq'} + G_{;}{}^{p'}\, G_{;q}{}^{q}{}_{p'}\right) \\
& + R\, G_{;p'}\, G_{;}{}^{p'} + R'\, G_{;p}\, G^{;p} \\
& - 2\, \left(R\, G_{;p'}{}^{p'} + R'\, G_{;p}{}^{p}\right) G + \frac{1}{2} R\, R' G^2 \ .
\end{aligned} \tag{2c}$$

Primes on indices denote tensor indices at the point x' and unprimed ones denote indices at the point x. R_{ab} and $R_{c'\,d'}$ are the Ricci tensor evaluated at the points x and x', respectively; R and R' are the scalar curvature evaluated at x and x'.

As shown in [20], under a conformal transformation of the metric,

$$\tilde{g}_{ab}(x) = \Omega(x)^2 g_{ab}(x) \ , \tag{3}$$

the noise kernel for a conformally invariant scalar field transforms as

$$\widetilde{N}_{abc'd'}(x,x') = \Omega(x)^{-2} N_{abc'd'}(x,x') \Omega(x')^{-2} \ . \tag{4}$$

Using this result in conjunction with (2) evaluated for the Minkowski vacuum state, we have computed an exact expression for the noise kernel for the conformally invariant scalar field in the conformal vacuum state of a large class of conformally flat spacetimes. As previously noted, these results are useful for those metrics that are conformal to the full Minkowski metric (or at least a part of it that contains a Cauchy surface).

The Wightman function in flat space in the Minkowski vacuum is

$$G^{+}(x,x') = \frac{1}{8\pi^2 \sigma(x,x')} + \frac{i}{8\pi} \delta(\sigma(x,x')) \mathrm{sgn}(t-t') \ , \tag{5}$$

with

$$\sigma(x,x') = \frac{1}{2} \left[-(x^0 - x^{0\,\prime})^2 + (\vec{x} - \vec{x}^{\,\prime})^2 \right] \ . \tag{6}$$

In general σ satisfies the relationship

$$\sigma = \frac{1}{2} g_{ab} \sigma^a \sigma^b = \frac{1}{2} g_{a'b'} \sigma^{a'} \sigma^{b'} \ , \tag{7}$$

with

$$\begin{aligned}
\sigma^a \ &\equiv \sigma^{;a} \ , \\
\sigma^{a'} &\equiv \sigma^{;a'} \ .
\end{aligned} \tag{8}$$

If the points are close together then

$$\begin{aligned}
\sigma^a \ &= x^a - x'^{\,a} + O[(x^a - x^{a'})^2] \ , \\
\sigma^{a'} &= \, - \, (x^a - x'^{\,a}) + O[(x^a - x^{a'})^2]
\end{aligned} \tag{9}$$

and of course in general

$$\begin{aligned}\sigma_a &= g_{ab}\sigma^b \\ \sigma_{a'} &= g_{a'\,b'}\sigma^{b'}\ .\end{aligned} \tag{10}$$

Because the noise kernel is quadratic in the Wightman function, one would expect to see formally divergent terms in the coincidence limit and for null separations which go like derivatives of $i\delta(\sigma)/\sigma$ and $\delta(\sigma)^2$. Using the symmetries of the noise kernel it is easy to show that all of the $i\delta(\sigma)/\sigma$ terms must vanish identically; however, the meaning of the $\delta(\sigma)^2$ terms are more difficult to understand.

As discussed in [12] in flat space and in [11] for de Sitter, these divergences may be more easily analyzed in Fourier space by integration against a test function; the resulting expressions are explicitly finite. The square of the Wightman function is

$$G^+(x,x')^2 = \frac{1}{64\pi^4\sigma^2} + \frac{i\delta(\sigma)\mathrm{sgn}(t-t')}{64\pi^3\sigma} - \frac{\delta(\sigma)^2}{64\pi^2}\ . \tag{11}$$

When integrated against a test function $f(x')$, we find that the divergent $f(x)\delta(0)$ and $if(x)/0$ terms arising from the integrals over $f(x')\delta(\sigma)^2$ and $if(x')\delta(\sigma)/\sigma$ are exactly the terms needed to cancel the divergence arising from the integral over $1/\sigma^2$ such that the total result is finite. As a result, the total integral is just the Hadamard finite part,

$$\int d^4x' f(x')[G^+(x,x')]^2 = \mathcal{H}\left[\int d^4x' \frac{f(x')}{64\pi^4\sigma(x,x')^2}\right]\ . \tag{12}$$

Therefore, the square of the Wightman function is a well-defined distribution.

A similar cancellation happens for the noise kernel. The resulting expression for the noise kernel for the conformally invariant scalar field in the Minkowski vacuum state is

$$\begin{aligned}\widetilde{N}_{abc'd'}(x,x') = &\Big[\frac{\sigma_a\sigma_b\sigma_{c'}\sigma_{d'}}{48\pi^4\sigma^6} + \frac{\sigma_{(a}\eta_{b)(c'}\sigma_{d')}}{24\pi^4\sigma^5} \\ &+\frac{4\eta_{a(c'}\eta_{d')b} - \eta_{ab}\eta_{c'd'}}{192\pi^4\sigma^4} \\ &-\frac{\sigma_a\sigma_b\sigma_{c'}\sigma_{d'}}{576\pi^2}\left(9\delta''(\sigma)^2 - 8\delta'(\sigma)\delta'''(\sigma) + \delta(\sigma)\delta''''(\sigma)\right) \\ &+\frac{\sigma_a\sigma_b\eta_{c'd'} + \eta_{ab}\sigma_{c'}\sigma_{d'}}{576\pi^2}\left(5\delta'(\sigma)\delta''(\sigma) - \delta(\sigma)\delta'''(\sigma)\right) \\ &+\frac{\sigma_{(a}\eta_{b)(c'}\sigma_{d')}}{144\pi^2}\delta(\sigma)\delta'''(\sigma) \\ &-\frac{\eta_{a(c'}\eta_{d')b}}{288\pi^2}\left(4\delta'(\sigma)^2 + \delta(\sigma)\delta''(\sigma)\right) \\ &+\frac{\eta_{ab}\eta_{c'd'}}{576\pi^2}\left(\delta'(\sigma)^2 - \delta(\sigma)\delta''(\sigma)\right)\Big] \\ = \mathcal{H}&\left[\frac{\sigma_a\sigma_b\sigma_{c'}\sigma_{d'}}{48\pi^4\sigma^6} + \frac{\sigma_{(a}\eta_{b)(c'}\sigma_{d')}}{24\pi^4\sigma^5} + \frac{4\eta_{a(c'}\eta_{d')b} - \eta_{ab}\eta_{c'd'}}{192\pi^4\sigma^4}\right]\ ,\end{aligned} \tag{13}$$

where the divergences arising from an integral over the delta functions are exactly those needed to cancel the divergences in the integration over powers of $1/\sigma$. Here (...) indicates symmetrization of the indices, η_{ab} is the Minkowski metric, and $\eta_{ac'} =$

diag$(-1, 1, 1, 1)$ is the bivector of parallel transport in Minkowski space for Cartesian coordinates. Combining (13) with (4) gives us

$$\widetilde{N}_{abc'd'}(x, x') = \mathcal{H}\left[\Omega(x)^{-2}\Omega(x')^{-2}\left(\frac{\sigma_a\sigma_b\sigma_{c'}\sigma_{d'}}{48\pi^4\sigma^6} + \frac{\sigma_{(a}\eta_{b)(c'}\sigma_{d')}}{24\pi^4\sigma^5} + \frac{4\eta_{a(c'}\eta_{d')b} - \eta_{ab}\eta_{c'd'}}{192\pi^4\sigma^4}\right)\right] . \tag{14}$$

This result agrees with previous computations of the noise kernel in the Minkowski vacuum state [19, 12, 20] and, as is shown in the appendix, with results by Osborn and Shore [23] and Cho and Hu [24] for the noise kernel in dS and AdS spacetimes.

3. The noise Kernel in de Sitter space

The noise kernel for the conformally invariant scalar field in the conformal vacuum in de Sitter space can be obtained from (14) with the substitutions

$$\begin{aligned} t &\to -\eta \\ t' &\to -\eta' \\ \Omega(x) &= \frac{\alpha}{(-\eta)} \\ \Omega(x') &= \frac{\alpha}{(-\eta')} , \end{aligned} \tag{15}$$

where we have represented de Sitter space in terms of the comoving coordinates with metric

$$ds^2 = \frac{\alpha^2}{(-\eta)^2}(-d\eta^2 + dx^2 + dy^2 + dz^2) . \tag{16}$$

However, we are primarily interested in investigating the noise kernel in the static coordinates, for which the metric has a form similar to that of Schwarzschild spacetime. The coordinate transformation to this system is

$$\begin{aligned} x &\equiv \frac{e^{-T/\alpha}}{\sqrt{B}}\rho\sin\theta\cos\phi , \\ y &\equiv \frac{e^{-T/\alpha}}{\sqrt{B}}\rho\sin\theta\sin\phi , \\ z &\equiv \frac{e^{-T/\alpha}}{\sqrt{B}}\rho\cos\theta , \\ -\eta &\equiv \alpha\frac{e^{-T/\alpha}}{\sqrt{B}} , \end{aligned} \tag{17}$$

and the resulting line element is

$$ds^2 = -BdT^2 + \frac{d\rho^2}{B} + \rho^2 d\theta^2 + \rho^2\sin^2\theta d\phi^2 , \tag{18}$$

with $B = 1 - \frac{\rho^2}{\alpha^2}$. For an observer situated at the origin, $B = 0$ is a cosmological horizon which marks the boundary of his observable universe. It is this coordinate system which interests us the most, since it provides an opportunity to study the noise kernel near the horizon and compare its behavior with that found for the approximate noise kernel in Schwarzschild spacetime.

From the definitions given in (17), we transform (14) to the static coordinates using the relation

$$N_{abc'd'}(x,x') = \frac{\partial x^A}{\partial x^a}\frac{\partial x^B}{\partial x^b}\frac{\partial x^{C'}}{\partial x^{c'}}\frac{\partial x^{D'}}{\partial x^{d'}} N_{ABC'D'}(x,x')\ , \tag{19}$$

where we have used capital letters to represent indicies of the comoving coordinates and lowercase to represent indicies of the static coordinates.

To avoid coordinate singularities, we express the noise kernel in terms of an orthonormal frame at each of the two points. We do this by introducing orthonormal basis vectors at each point which satisfy

$$(e_{\hat{a}})^c (e_{\hat{b}})_c = \eta_{\hat{a}\hat{b}}\ , \tag{20}$$

$$(e^{\hat{a}})_c (e_{\hat{a}})_d = g_{cd}\ . \tag{21}$$

Here $\eta_{\hat{a}\hat{b}}$ is the Minkowski metric. The components of a vector may be written in the orthonormal basis as

$$A_{\hat{a}} = (e_{\hat{a}})^a A_a\ . \tag{22}$$

Similarly, the noise kernel in this basis is

$$N_{\hat{a}\hat{b}\hat{c}'\hat{d}'}(x,x') = (e_{\hat{a}})^a (e_{\hat{b}})^b (e_{\hat{c}'})^{c'} (e_{\hat{d}'})^{d'} N_{abc'd'}(x,x')\ . \tag{23}$$

For the static de Sitter coordinates, we choose basis vectors such that

$$\begin{aligned}
(e_{\hat{T}})^T &= \sqrt{-g^{TT}} \\
(e_{\hat{\rho}})^\rho &= \sqrt{g^{\rho\rho}} \\
(e_{\hat{\theta}})^\theta &= \sqrt{g^{\theta\theta}} \\
(e_{\hat{\phi}})^\phi &= \sqrt{g^{\phi\phi}}\ .
\end{aligned} \tag{24}$$

All other components are zero.

In general, the expressions resulting from this procedure are quite long. Although we have computed every component for the noise kernel in the static coordinates, for the sake of brevity we present only two of them:

$$\begin{aligned}
N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x,x') &= (BB')^{-1} N_{TTT'T'}(x,x') \\
&= \mathcal{H}\left[\frac{1}{12\pi^4\left[\alpha^2\left(\sqrt{BB'}\,\tau-2\right)+2\rho\rho'\cos(\gamma)\right]^6}\right. \\
&\quad \times\Big\{\alpha^4\left[-12\sqrt{BB'}\,\tau + BB'\left(\tau^2+14\right)\right. \\
&\quad \left.-\left(2B+2B'-6\right)\left(\tau^2-1\right)\right] \\
&\quad +4\alpha^2\rho\rho'\cos(\gamma)\left(3\sqrt{BB'}\,\tau-2\left(\tau^2-1\right)\right) \\
&\quad \left.+2\rho^2\rho'^{\,2}\left(\tau^2-1\right)\cos(2\gamma)\Big\}\right]
\end{aligned} \tag{25a}$$

$$
\begin{aligned}
N_{\hat{T}\hat{\rho}\hat{T}'\hat{\rho}'}(x,x') &= N_{T\rho T'\rho'}(x,x') \\
&= \mathcal{H}\left[\frac{\alpha^2}{6\pi^4\left[\alpha^2\left(\sqrt{BB'}\,\tau-2\right)+2\rho\rho'\cos(\gamma)\right]^6}\right. \\
&\quad \times\left\{\alpha^2\cos(\gamma)\left[-4\sqrt{BB'}\,\tau+BB'\left(\tau^2+4\right)\right.\right. \\
&\quad \left.-\left(2B-2B'+2\right)\left(\tau^2-2\right)\right] \\
&\quad \left.\left.+\rho\rho'(\cos(2\gamma)+3)\left(\sqrt{BB'}\,\tau-\tau^2+2\right)\right\}\right]\ , \qquad (25b)
\end{aligned}
$$

where

$$
\begin{aligned}
\tau &\equiv 2\cosh(\Delta T/\alpha)\ , && (25c)\\
\cos\gamma &\equiv \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\ , && (25d)
\end{aligned}
$$

and $B' = 1 - \rho'^{\,2}/\alpha^2$.

The particular form of the orthonormal frame used here was chosen to facilitate comparison with [20], which used an equivalent orthonormal frame to evaluate the noise kernel in Schwarzschild spacetime. However, it is important to note that at the horizon T is a null coordinate and is infinite, and that the transformation to the orthonormal frame given in (23) is similarly ill-defined. Therefore, although (25a) and (25b) remain valid arbitrarily close to the horizon, they should not be considered valid at the horizon itself.

3.1. Behavior near the horizon

When either of the two points approaches the cosmological horizon, we see that $\rho \to \alpha$ and $B \to 0$ (or $\rho' \to \alpha$ and $B' \to 0$, respectively). For finite ΔT, inspection of (25) shows that both of the components displayed therein remain bounded when either point is arbitrarily close to the horizon, so long as the points are spacelike or timelike separated.$^+$

For example, if we take ρ' to be arbitrarily close to the horizon, then

$$
\begin{aligned}
N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x,x') &\approx -\frac{\tau^2-1}{384\pi^4\alpha^4(\alpha-\rho\cos(\gamma))^6} \\
&\quad\times\left[\alpha^2(B-3)+4\alpha\rho\cos(\gamma)-\rho^2\cos(2\gamma)\right] && (26a)\\
N_{\hat{T}\hat{\rho}\hat{T}'\hat{\rho}'}(x,x') &\approx -\frac{\tau^2-2}{384\pi^4\alpha^3(\alpha-\rho\cos(\gamma))^6} \\
&\quad\times\left[(2B-4)\alpha\cos(\gamma)+\rho(\cos(2\gamma)+3)\right]\ . && (26b)
\end{aligned}
$$

When $\rho = \rho'$ and both points are near the horizon, we have

$$
N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x,x') \approx \frac{\tau^2-1}{3072\pi^4\alpha^8}\csc^8\left(\frac{\gamma}{2}\right)\ . \qquad (27a)
$$

$^+$ We find this behavior to be true for all components of the noise kernel when expressed in the orthonormal frame.

and

$$N_{\hat{T}\hat{\rho}\hat{T}'\hat{\rho}'} \approx -\frac{\tau^2-2}{3072\pi^4\alpha^8}\csc^8\left(\frac{\gamma}{2}\right) . \tag{27b}$$

Therefore, the noise kernel is bounded when either or both of the points are near the horizon so long as the separation in the T coordinate is finite and either $\gamma \neq 0$, $\rho \neq \rho'$, or both. For $\gamma = 0$ and $\rho = \rho'$, the above expression is not bounded as the two points approach the horizon.

However, although the expressions given in (25) remain valid arbitrarily close to the horizon, one must take care when attempting to evaluate the noise kernel at the horizon itself. This is because T is ill-defined on the horizon, and therefore (19) no longer holds. On the other hand, there is nothing per se wrong with taking the near horizon limit (the orthonormal frame remains perfectly well defined except at the horizon itself) as long as one is clear about what is happening physically to the T coordinate in that limit.

There are two ways to consider taking the near horizon limits which lead to different results. The first is to consider what happens to τ on the future (or past) horizon, where we take $\rho = \rho' \to \alpha$, $T \to \pm\infty$, and $T' \to \pm\infty$. In this limit we can obtain a well-defined expression for τ by considering either the comoving coordinates (for the future horizon) or null de Sitter-Kruskal coordinates (for both the future and past horizons).

In the comoving coordinates, we have

$$-\eta = \alpha\frac{e^{-T/\alpha}}{\sqrt{B}} , \qquad -\eta' = \alpha\frac{e^{-T'/\alpha}}{\sqrt{B'}} . \tag{28}$$

Holding η and η' constant in the limit that $\rho = \rho' \to \alpha$,

$$\Delta T = \alpha\ln(\eta'/\eta) , \tag{29}$$

and

$$\tau = \frac{\eta'}{\eta} + \frac{\eta}{\eta'} . \tag{30}$$

Both η and η' are well defined on the future horizon, so these are well defined quantities.

The expression is similar in null de Sitter-Kruskal coordinates. In these coordinates [36],

$$u = e^{T/\alpha}\sqrt{\frac{\alpha-\rho}{\alpha+\rho}} , \qquad v = e^{-T/\alpha}\sqrt{\frac{\alpha-\rho}{\alpha+\rho}} . \tag{31}$$

Again holding u and u' constant while taking $\rho = \rho' \to \alpha$, we have $v = v' \to 0$ on the future horizon,

$$\Delta T = \alpha\ln(u/u') , \tag{32}$$

and

$$\tau = \frac{u}{u'} + \frac{u'}{u} \tag{33}$$

are well defined.

As a result, it is acceptable to consider (27) on the future horizon; although both T and T' are ill-defined in this limit, ΔT is still a meaningful quantity. τ depends on the separation between points in this limit, with $\tau = 2$ when the points come together.

A similar analysis holds in the limit that the two points are taken to the past horizon. In this case, we take $\rho = \rho' \to \alpha$ and $u = u' \to 0$ while holding v and v' constant. The result is

$$\tau = \frac{v}{v'} + \frac{v'}{v} , \tag{34}$$

which is once again well defined.

On the other hand, if we hold T and T' constant as we take $\rho = \rho' \to \alpha$, this corresponds to points on the bifurcation surface. This is problematic, since τ may be specified arbitrarily by choosing values of ΔT despite the fact that all such choices correspond to the same limit points on the bifurcation surface; for zero angular separation the noise kernel will diverge, whereas for non-zero separation the noise kernel will take on finite but arbitrary values.

We can demonstrate this behavior in the comoving coordinates by noting that both η and η' go to $-\infty$, as does $\Delta\eta$ when $T \neq T'$. On the other hand, the ratio

$$\frac{\eta'}{\eta} = e^{\Delta T/\alpha} \tag{35}$$

is finite but arbitrary.

Likewise in the de Sitter-Kruskal coordinates, we have

$$\tau = \sqrt{\frac{uv'}{u'v}} + \sqrt{\frac{u'v}{uv'}} , \tag{36}$$

with u, u', v and v' going to zero on the bifurcation surface. Again, this leads to arbitrary values for τ depending on how the limit is taken. This indicates that the bifurcation surface is not a proper limit for the noise kernel when expressed in the static coordinates and the orthonormal frame (24).

Finally, we note that τ diverges in the case where one point is on the past horizon and the other is on the future horizon, or the case where one point is on the bifurcation surface and the other is on the future or past horizon, since $|\Delta T| \to \infty$ in each of these cases.

3.2. Comparison with Schwarzschild spacetime

In [20], an approximate expression for the noise kernel was computed for the conformally invariant scalar field in Schwarzschild spacetime. Explicit expressions were given for the $N^{t}{}_{t}{}^{t'}{}_{t'}$ component of noise kernel in the case that the separation was only in time or only in terms of the radial coordinate r.

As discussed in Sec. 1, we are interested in comparing the noise kernel obtained for Schwarzschild spacetime with that obtained for de Sitter space in the static coordinates. We do this by generating an expansion for the noise kernel in the static de Sitter coordinates that is equivalent to the quasi-local expansion used for the Schwarzschild case. In principle, the proper way to do this is to use a quasi-local expansion for the Wightman function in the static de Sitter coordinates and recompute the noise kernel using that expression. However, since we are primarily interested in comparing our results with (4.16) and (4.17) of [20], we can generate equivalent approximations by expanding (25a) in powers of $1/\Delta T$, $1/\Delta\rho$, and $\eta \equiv \cos\gamma - 1$ and truncating the series at the appropriate order.

For our investigation, we consider the $N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}$ component and begin by splitting in the time and radial directions. The result is

$$[N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}]_{\Delta\rho=\gamma=0} = \frac{1}{4\pi^4\alpha^8 B^4(\tau-2)^4}\,, \tag{37a}$$

$$[N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}]_{\Delta T=\gamma=0} = -\frac{B+B'-2BB'+2(\sqrt{BB'}-1)\widetilde{B}}{64\pi^4\alpha^8(\sqrt{BB'}-\widetilde{B})^6}\,. \tag{37b}$$

Here, $\widetilde{B} \equiv 1 - \rho\rho'/\alpha^2$.

Expanding the $N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}$ component in powers of $1/\Delta T$ and $1/\Delta\rho$ and truncating the series at order $O[(x-x')^{-4}]$, we find

$$[N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x,x')]_{\text{series}} = \frac{1}{4\pi^4 B^4}\left[\frac{1}{\Delta T^8} - \frac{1}{3\alpha^2\Delta T^6} + \frac{7}{120\alpha^4\Delta T^4}\right] + O[\Delta T^{-2}]\,, \tag{38a}$$

$$\begin{aligned}[N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x,x')]_{\text{series}} = \frac{1}{\pi^4}\Bigg[&\frac{B^4}{4\Delta\rho^8} - \frac{(B-1)B^3}{\rho\Delta\rho^7} + \frac{(5-6B)B^2}{4\alpha^2\Delta\rho^6} \\ &+ \frac{(2B^2-3B+1)\,B}{2\alpha^2\rho\Delta\rho^5} + \frac{8B^2-8B+1}{32\alpha^4\Delta\rho^4}\Bigg] \\ &+ O[\Delta\rho^{-3}]\,. \end{aligned}\tag{38b}$$

In [20], we considered thermal states in which $\kappa \equiv T/2\pi$ was left arbitrary. However, the natural choice of state in Schwarzschild spacetime is the Hartle-Hawking state [34], for which $\kappa = 1/4M$, since only in this state does the stress-energy tensor remain finite at the horizon; for all other thermal states, including the zero-temperature Boulware state [38], the stress-energy tensor diverges badly at $r = 2M$. We may compare the above expressions with those of [20] by setting $\kappa = 1/4M$ and taking $B \to f = 1 - 2M/r$ and $\alpha \to 1/\kappa$. In general, we find that although the coefficients at each order are different, the general form of the expansion is the same. In addition, for time separations we find that to leading order in powers of $1/f$ the two expressions are identical. For radial separations, the situation is slightly different; in the near horizon limit the only surviving term is proportional to $\kappa^4/\Delta r^4$ for Schwarzschild and $1/\alpha^4\Delta\rho^4$ for de Sitter. Thus, the expressions are equal up to a rescaling of $\Delta\rho$. This suggests that the behavior of the full noise kernel in de Sitter space can tell us something significant about the behavior of the full noise kernel in the near horizon region in Schwarzschild and provides evidence for our conjecture that the validity of the short distance expansion for the two metrics should be similar (at least for separations in the time or radial directions).

In [20], we noted that we expected the quasi-local expansion to be valid when the geodesic distance was small in comparison to the mass scale of the black hole in the region near the horizon. For de Sitter space, the corresponding distance scale is given by the Hubble parameter α. To investigate the range of validity of the expressions above,

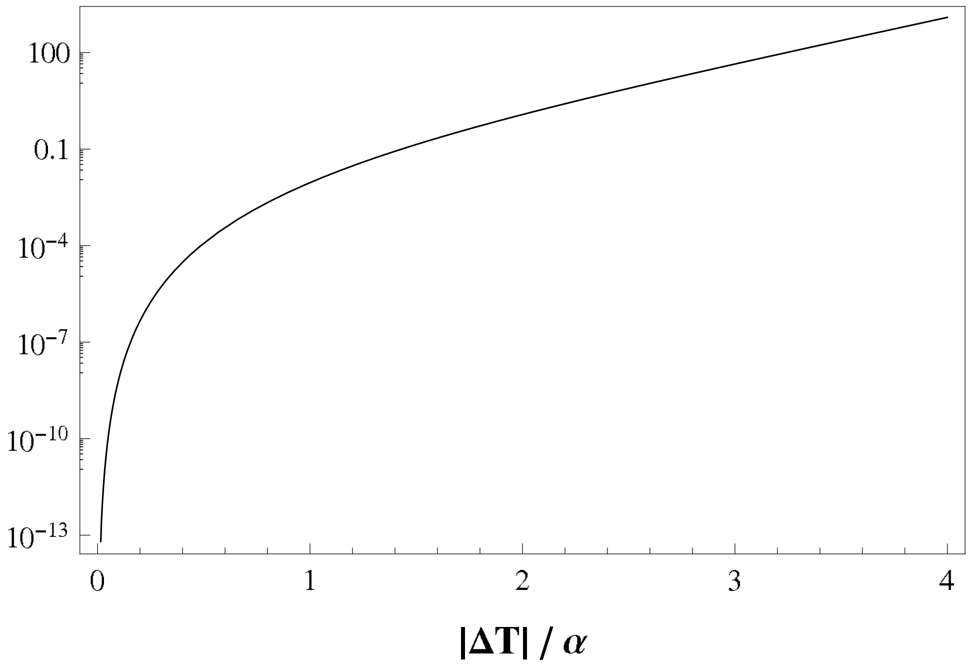

Figure 1. This figure shows the relative error in (39) due to separation in T, with $\Delta\rho = \gamma = 0$. This error is independent of the distance from either point to the cosmological horizon.

we compute the relative error between the exact expression for the noise kernel and the truncated series expansions,

$$\left| \frac{[N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x,x')]_{\text{series}}}{N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x,x')} - 1 \right| . \tag{39}$$

For the purposes of our investigation, we define the region of validity to be the region within which the relative error is less than 10%.

When the points are separated in time, we find that the error between the truncated series and the exact expression for static de Sitter space is approximately 10% when $|\Delta T| \approx 1.5\alpha$ (see figure 1). This result is independent of the distance between the two points and the horizon, as can be seen by inspection of (37a) and (38a). For radial separation, the situation is somewhat different. Unlike in Schwarzschild spacetime, where we are interested in points outside the event horizon, in the static de Sitter case we are interested in the region inside the cosmological horizon; thus, the radial separation can never be larger than α without one point crossing the horizon or the origin. What we find is that the error remains small so long as both points are sufficiently far from the horizon. As either point nears the cosmological horizon, we find that region of validity scales roughly linearly with the distance from the horizon. Figures 2 - 3 illustrate this behavior.

In contrast with the cases of time and radial separation, this type of analysis fails when the points are separated in only the angular direction. The reason for this is that when we expand (25a) in powers of $1/\eta$ with $\Delta T = \Delta\rho = 0$, we find that the resulting series contains only terms up to order $O[(x-x')^{-4}]$ and thus provides an exact

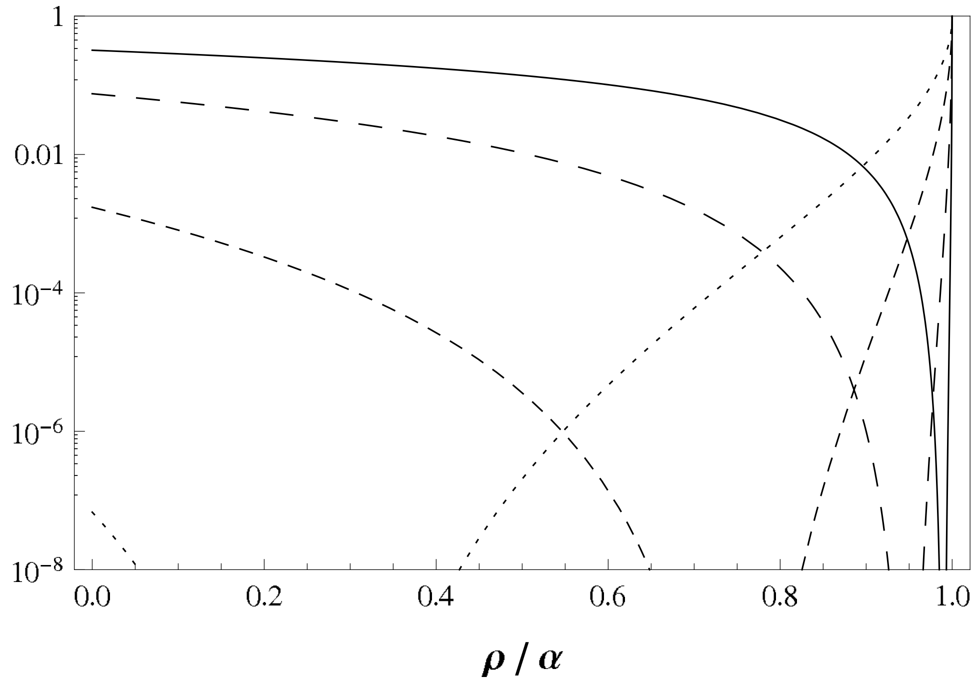

Figure 2. This figure shows the relative error in (39) due to changes in ρ for $\rho' = 0.99\alpha$ (solid line), 0.95α (long dashes), 0.75α (medium dashes), and 0.25α (short dashes) with $\Delta T = \gamma = 0$. On this scale, $\rho/\alpha = 1$ marks the cosmological horizon.

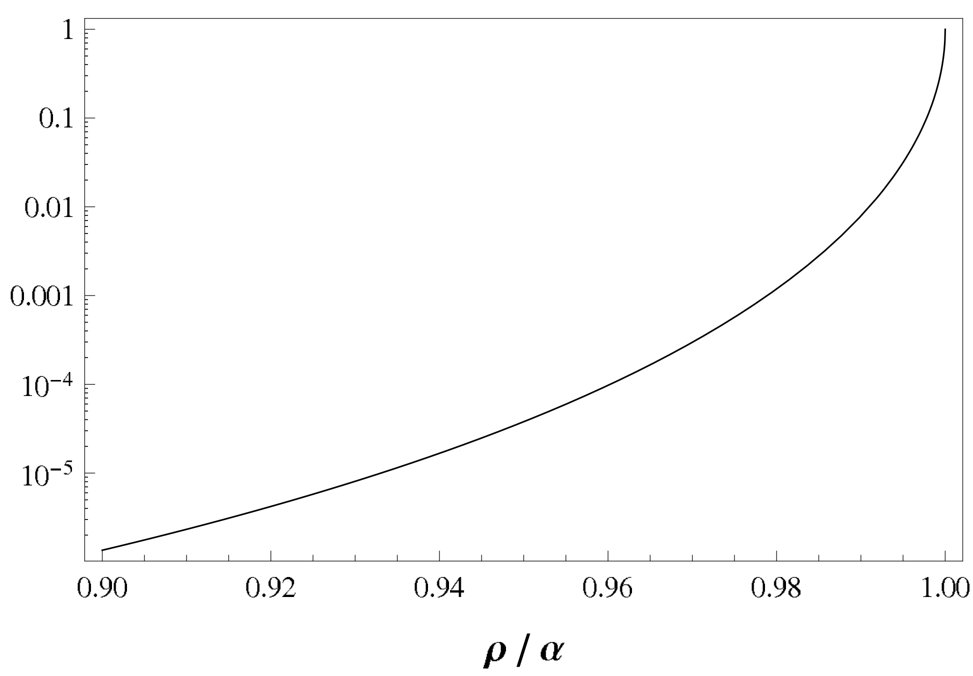

Figure 3. This figure shows the relative error in (39) due to fixed radial separations of $\Delta\rho = 0.1\alpha$ as both points near the horizon, with $\Delta T = \gamma = 0$. On this scale, $\rho/\alpha = 1$ marks the cosmological horizon.

expression for the noise kernel. Explicitly, we find

$$\begin{aligned}[N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x,x')]_{\text{series}} &= -\frac{1}{128\pi^4\eta^6\rho^8} - \frac{1}{64\pi^4\eta^5\rho^8} + \frac{1}{128\pi^4\eta^4\rho^8} \\ &= N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x,x') \; . \end{aligned} \tag{40}$$

However, we suspect that this behavior is a coincidence due to the symmetries present in de Sitter space and that this result will likely not hold for Schwarzschild spacetime.

4. Discussion

We have computed an expression for the noise kernel for the conformally invariant scalar field in any metric that is conformal to the full Minkowski metric (or at least enough of it to contain a Cauchy surface) when the field is in the conformal Minkowski vacuum state. A general form for this expression is shown in (14).

Using the above result, we have computed an exact expression for the noise kernel in the conformal Minkowski vacuum (which is the same as the Bunch-Davies state) in de Sitter spacetime. All components have been computed for an arbitrary separation of the points; however, for brevity only two components are displayed in (25).

Investigating these expressions in the region near the cosmological horizon, we found that all components of the noise kernel remained finite so long as the points are not null-separated. However, because T is ill-defined on the horizon, one must take care when taking the near-horizon limit of the noise kernel. For points on the future horizon this limit is well-defined; however, on the bifurcation surface the noise kernel may take on arbitrary values when expressed in the static coordinates in the orthonormal frame given in (24), so this should not be considered a proper limit.

Since the metric for the static de Sitter coordinates is similar in form to the metric for Schwarzschild spacetime, with the event horizon replaced by the cosmological horizon for observers at the origin, it is expected that the behavior of the noise kernel near the cosmological horizon in the static de Sitter coordinates should be similar to the behavior of the noise kernel near the event horizon in Schwarzschild spacetime where only an approximate solution has been obtained [20]. In the static de Sitter coordinate system, the Bunch-Davies state that we use is equivalent to a Gibbons-Hawking state [33] with $\kappa = 1/4\alpha$; as a result, the noise kernel for this state may be compared with the noise kernel for Schwarzschild spacetime when the field is in the Hartle-Hawking state [34], with the Hubble distance α playing the role of the mass M. Thus, (25) may be compared directly with the approximate expression for the noise kernel in Schwarzschild spacetime obtained in [20], where an expansion in terms of inverse powers of the geodesic separation was used. To do this, we obtained a similar expression for the noise kernel for the static de Sitter coordinates by expanding the expression for the $N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}$ component in (25a) in powers of ΔT and $\Delta\rho$. As expected, we found that the leading order behaviors of both expansions are identical, and that the remaining orders are similar in form, although the coefficients are different.

It is also of interest to investigate the range of validity of the quasi-local approximation used in [20]. To do so, we compared the exact result for the noise kernel computed in the static de Sitter coordinates with the truncated series expansion. In general, we found that the quasi-local approximation remains valid for time separations smaller than the Hubble distance α, and found an error of approximately 10% when $\Delta T \approx 1.5\alpha$. For radial separations, we are restricted to separations no greater than α. We found that the relative error remains small so long as both points are sufficiently far from the cosmological horizon, but goes to infinity as either point approaches the horizon. When both points are near the horizon, we find that the region of validity scales roughly linearly with the distance between the point nearest the horizon and the horizon itself. Finally, for angular separations, we find that the expression generated by the expansion procedure is equal to the exact expression if all three terms are kept; however, we suspect that this result is an artifact of the symmetry of de Sitter space and will not hold in Schwarzschild spacetime.

From these results, we expect the quasi-local approximation used in [20] to be valid when the separation of the points is less than the mass scale, so long as neither point is too close to the horizon. In addition, we expect for radial separations that the range of validity should scale roughly linearly with distance from the horizon. Finally, if one or both points are on the horizon, we expect the quasi-local approximation to be invalid; however, based on our results in static de Sitter space we expect that the exact noise kernel for Schwarzschild spacetime will remain finite on the horizon so long as the two points are not null-separated.

The comparison made here between the noise kernel in static de Sitter spacetime with respect to the Gibbons-Hawking vacuum and that in the Schwarzschild spacetime with respect to the Hartle-Hawking vacuum, both in the near-horizon region, is useful for a study of the backreaction of Hawking radiation on a quasi-static black hole enclosed in a box and the behavior of the quantum field-induced metric fluctuations via the Einstein-Langevin equation. It is our hope to explore further along this direction in subsequent papers.

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Appendix A. The noise kernel for maximally symmetric spacetimes.

In maximally symmetric spacetimes such as Minkowski space, de Sitter space or anti-de Sitter space, there is an alternative representation for bi-tensors like the noise kernel in terms of a set of purely geometric objects: the geodesic separation biscalar $d(x, x')$, its derivatives with respect to each point $d_a \equiv \nabla_a d(x, x')$ and $d_{c'} \equiv \nabla_{c'} d(x, x')$, the metric, and the bivector of parallel transport $g_{ac'}(x, x')$ [39]. In Minkowski space, these objects are given by

$$\begin{aligned} d(x, x') &= \sqrt{-(x^0 - x^{0\prime})^2 + (\vec{x} - \vec{x}')^2} \\ d_a &= (x_a - x'_a) \\ g_{ac'} &= diag(-1, 1, 1, 1) \ . \end{aligned} \tag{A.1}$$

Due to the symmetries involved there are five such terms in the expression for the noise kernel,

$$\begin{aligned} N_{abc'd'}(x, x') = C_1 n_a n_b n_{c'} n_{d'} &+ C_2 (g_{ab} n_{c'} n_{d'} + n_a n_b g_{c'd'}) \\ &+ 4C_3 n_{(a} g_{b)(c'} n_{d')} + 2C_4 g_{a(c'} g_{d')b} + C_5 g_{ab} g_{c'd'} \end{aligned} \tag{A.2}$$

where the coefficients C_1 through C_5 are functions only of the geodesic separation $d(x, x')$.

Using this representation, Osborn and Shore [23] computed the coefficients of the stress-energy bi-tensor for free scalar fields in Euclidean spacetimes of constant curvature. Cho and Hu [24] have also computed them for scalar fields with arbitrary mass and coupling in the Euclidean sector of anti-de Sitter space. As a check on our (14), we would like to rewrite that expression in the form of (A.2).

For the Minkowski vacuum the coefficients may be read off directly from (14) by noting that $d(x, x') = \sqrt{2\sigma(x, x')}$ and $g_{ac'} = -\sigma_{ac'}$. We find that

$$\begin{aligned} C_1 &= \frac{4}{3\pi^4 d(x, x')^8} \\ C_2 &= 0 \\ C_3 &= \frac{1}{3\pi^4 d(x, x')^8} \\ C_4 &= \frac{1}{6\pi^4 d(x, x')^8} \\ C_5 &= -\frac{1}{12\pi^4 d(x, x')^8} \ . \end{aligned} \tag{A.3}$$

For de Sitter space, the comparison is made by considering separations of points along the time direction and noting that in the cosmological coordinates $d(x, x')^2 = -(t - t')^2$ for time separations. Due to the symmetries of Euclidean de Sitter, it is possible to rewrite any arbitrary separation as a time separation by relabelling of coordinate axes, so it is sufficient to consider only time separations when making the comparison. We find that the coefficients are given by

$$\begin{aligned}
C_1 &= \frac{4}{3\pi^4 d(x,x')^8} - \frac{4}{9\pi^4\alpha^2 d(x,x')^6} + O(d(x,x')^{-4}) \\
C_2 &= 0 \\
C_3 &= \frac{1}{3\pi^4 d(x,x')^8} - \frac{1}{9\pi^4\alpha^2 d(x,x')^6} + O(d(x,x')^{-4}) \\
C_4 &= \frac{1}{6\pi^4 d(x,x')^8} - \frac{1}{18\pi^4\alpha^2 d(x,x')^6} + O(d(x,x')^{-4}) \\
C_5 &= -\frac{1}{12\pi^4 d(x,x')^8} + \frac{1}{36\pi^4\alpha^2 d(x,x')^6} + O(d(x,x')^{-4})\,.
\end{aligned} \tag{A.4}$$

For anti-de Sitter space the same method is used; however, the computation proceeds slightly differently due to the need to specify boundary conditions at infinity and the fact that anti-de Sitter space is conformally related to only half of Minkowski space [37]. Choosing Dirichlet boundary conditions, the appropriate vacuum Wightman function for Minkowski spacetime with a boundary located at $x = 0$ (and with points non-null separated) is given by

$$G^{+}(x,x') = \frac{1}{8\pi^2\sigma} - \frac{1}{8\pi^2\bar{\sigma}}\,, \tag{A.5}$$

where $\bar{\sigma} = \frac{1}{2}[-(t-t')^2 + (x+x')^2 + (y-y')^2 + (z-z')^2]$.

Plugging this expression into (2) and following the same procedure used for the de Sitter result, we find

$$\begin{aligned}
C_1 &= \frac{4}{3\pi^4 d(x,x')^8} - \frac{10}{9\pi^4\alpha^2 d(x,x')^6} + O(d(x,x')^{-4}) \\
C_2 &= \frac{1}{12\pi^4\alpha^2 d(x,x')^6} + O(d(x,x')^{-4}) \\
C_3 &= \frac{1}{3\pi^4 d(x,x')^8} - \frac{7}{36\pi^4\alpha^2 d(x,x')^6} + O(d(x,x')^{-4}) \\
C_4 &= \frac{1}{6\pi^4 d(x,x')^8} - \frac{5}{72\pi^4\alpha^2 d(x,x')^6} + O(d(x,x')^{-4}) \\
C_5 &= -\frac{1}{12\pi^4 d(x,x')^8} + \frac{1}{72\pi^4\alpha^2 d(x,x')^6} + O(d(x,x')^{-4})\,.
\end{aligned} \tag{A.6}$$

To test the results of the method used in Sec. 2, we compare these expressions against those computed by Osborn and Shore [23] and by Cho and Hu [24]. In [23], Osborn and Shore used geometric techniques to derive an exact expression for the stress-energy bi-tensor for Euclideanized de Sitter and anti-de Sitter space. To compare their expression with ours, we first expand it in terms of the geodesic distance, and then analytically continue to the Lorentzian sector. Although we display only up through order $O(d(x,x')^{-6})$, we have evaluated the expansions in (A.4) and (A.6) up through $O(d(x,x')^{10})$ and find agreement with Osborn and Shore at all orders tested.

In [24], Cho and Hu used a different approach - the zeta function method - to compute an approximate expression for the stress-energy bi-tensor in Euclideanized anti-de Sitter. For short separations, they derived an expression in terms of the geodesic distance up to order $O(d(x,x')^{-6})$. Again, we find agreement between the expression they derived and (A.6) above.

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