Solution OF Elementary Problem
E 2558. Proposed By a. Torchinsky, Cornell University

Suppose that $\sum a_n$ is a divergent series of positive terms, let $s_n = a_1 + \cdots + a_n$ for n=1,2,…. For which values of p does the series $\sum a_n / s_n^p$ converge?

Solution by Elmer K. Hayashi. We prove a more general theorem from which we deduce that $\sum a_n / s_n^p$ if and only if $p > 1$.

Theorem. Let $f(x)$, for $x > 0$, be any nonnegative, continuous, monotonically decreasing, real-valued function. if $\sum a_n$ is a divergent series of positive terms and if $s_n = a_1 + \cdots + a_n$ for n=1,2,… , then

$$\sum a_n f(s_n) \text{ converges if } \int_{s_1}^{\infty} f(x)dx < \infty ,$$

and

$$\sum a_n f(s_{n-1}) \text{ diverges if } \int_{s_1}^{\infty} f(x)dx = \infty .$$

Proof:……