SOLUTION OF ELEMENTARY PROBLEM

E 2558. Proposed by A. Torchinsky, Cornell University

Suppose that $\sum a_n$ is a divergent series of positive terms, and let $s_n = a_1 + \cdots + a_n$ for $n = 1, 2, \ldots$. For which values of p does the series $\sum a_n/s_n^p$ converge?

Solution by Elmer K. Hayashi. We prove a more general theorem from which we deduce that $\sum a_n/s_n^p$ converges if and only if $p > 1$.

Theorem. Let $f(x)$, for $x > 0$, be any nonnegative, continuous, monotonically decreasing, real-valued function. If $\sum a_n$ is a divergent series of positive terms and if $s_n = a_1 + \cdots + a_n$ for $n = 1, 2, \ldots$, then

$$\sum a_n f(s_n) \quad \text{converges if} \quad \int_{s_1}^{\infty} f(x)\, dx < \infty,$$

and

$$\sum a_n f(s_{n-1}) \quad \text{diverges if} \quad \int_{s_1}^{\infty} f(x)\, dx = \infty.$$

Proof: Intuitively we reason that if $u = s_n$ then du is analogous to $s_n - s_{n-1} = a_n$. Hence $\sum a_n f(s_n)$ probably behaves somewhat like $\int f(u)du$. Furthermore, if $F(x)$ is any antiderivative of the continuous function $f(x)$, then $\int_a^b f(u)du = F(b) - F(a)$. Thus a natural series with which to compare $\sum a_n f(s_n)$ is the telescoping series

$$\sum_{n=2}^{\infty} \{F(s_n) - F(s_{n-1})\} \tag{1}$$

since

$$\sum_{k=2}^{n} \{F(s_k) - F(s_{k-1})\} = F(s_n) - F(s_1) = \int_{s_1}^{s_n} f(x)\, dx. \tag{2}$$

From equation (2), it is apparent that the series (1) converges if and only if the integral, $\int_{s_1}^{\infty} f(x)\, dx$, is convergent. Now, by the mean value theorem,

$$F(s_k) - F(s_{k-1}) = F'(C_k)(s_k - s_{k-1}) = a_k f(c_k)$$

for some c_k between s_{k-1} and s_k. Since f is monotonically decreasing, we have for $k = 2, 3, \ldots$,

$$F(s_k) - F(s_{k-1}) \le a_k f(s_{k-1})$$

and

$$F(s_k) - F(s_{k-1}) \ge a_k f(s_k).$$

Using the Comparison test, we arrive at the conclusion of the theorem.

If we take $f(x) = x^{-p}, p \ge 0$, we conclude that $\sum a_n/s_n^p$ converges for $p > 1$ and $\sum a_n/s_{n-1}^p$ diverges for $0 \le p \le 1$. In general, it is not true that if $\sum a_n f(s_{n-1})$ is diverent, then $\sum a_n f(s_n)$ is also divergent. For example, if $f(x) = \frac{1}{x \log x}$, $a_1 = s_1 = 1 + e$ and