# New Results On Composition-Delay Equations With Asymptotically Periodic Solutions

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## Introduction

**Definition 1 (Difference Equation)** *An $n^{th}$ order difference equation is an equation of the form*

$$y_i = f(y_{i-1}, \dots, y_{i-n}) \tag{1}$$

*where $f$ is some function (often continuous) mapping $Y^n \to Y$.*

**Definition 2** *A set $\{y_i\}$ is a solution to a difference equation given by $f$ if (**??**) holds.*

- Integer initial values

Examples:

- Linear equations: $y_n = y_{n-1} + y_{n-2}$ (Fibonaccci, Lucas numbers)
- Collatz Conjecture:

$$a_k = \begin{cases} \frac{a_{k-1}}{2} & \text{if } a_{k-1} \text{ is even} \\ 3a_{k-1} + 1 & \text{if } a_{k-1} \text{ is odd} \end{cases} \tag{2}$$

** Goal: Develop tools to understand behavior of non-linear, non-differentiable difference equations. **

## The Equation $y_n = \min\{y_{n-k_1} - y_{n-m_1}, y_{n-k_2} - y_{n-m_2}\}$

**Theorem 1** *Suppose $k, m, j \geq 1$ and consider solutions to the equation $y_n = y_{n-k} - y_{n-m}$, $n \geq 0$.*

1. *There exists a prime period $6j$ solution if and only if $(k, m) \in \{(j, 2j), (5j, 4j)\} \mod 6j$.*
2. *If $\gcd(k, m) = 1$ then all solutions are eventually periodic if and only if $(k, m) = (1, 2)$. Furthermore, if $(k, m) = (1, 2)$ then all non-trivial solutions must be strictly periodic with period six.*
3. *All solutions are either asymptotically periodic or satisfy*

$$\limsup_{n\to\infty} \frac{|y_n|}{n} = \infty. \tag{3}$$

**Theorem 2** *Suppose $k_1 = m_1 + k_2$ and $\gcd(k_1, m_1, k_2, m_2) = 1$.*

1. *There exists a non-trivial solution which has period $p = m_1 + m_2$.*
2. *All solutions are asymptotically periodic with period $p = m_1 + m_2$ if and only if $k_2 | m_2$.*
3. *If $k_2 \nmid m_2$ then there exists a non-trivial period $k_2$ solution.*
4. *If $\gcd(k_2, m_1 + m_2) > 1$, then there exists an unbounded integer solution which satisfies*

$$\limsup_{n\to\infty} \frac{|y_n|}{n} = \frac{1}{\operatorname{lcm}(k_2, m_1 + m_2)}. \tag{4}$$

5. *Any unbounded integer solution to the equation must satisfy*

$$\limsup_{n\to\infty} \frac{|y_n|}{n} < \infty. \tag{5}$$

**Other Results and Conjectures:**

The "absolute value case" $y_n = \min\{y_{n-k_1} - y_{n-m_2}, y_{k_2} - y_{m_2}\}$ was completely solved.

**Conjecture 1** *Suppose $\gcd\{k_1, k_2, m_1, m_2\} = 1$ and $k_1 + k_2 = m_1 = m_2 = m$. Then, all solutions to $y_n = \min\{y_{n-k_1} - y_{n-m}, y_{n-k_2} - y_{n-m}\}$ are strictly periodic, and moreover there exists a $Q > 1$ such that for all $q > Q$ there exists some solution with prime period $2q$.*

Majority of work discussed in K. S. Berenhaut, R. T. Guy, "Periodicity and boundedness for the integer solutions to a minimum-delay difference equation," in press. (*Journal of Difference Equations and Applications*)

**Abstract** The purpose of this thesis is to convey several new results in the field of piecewise difference equations, paying particular attention to higher order equations with asymptotically periodic solutions. A study of solutions to the equation

$$y_n = \min\{y_{n-k_1} - y_{n-m_1}, y_{n-k_2} - y_{n-m_2}\}$$

is presented. Related results are then obtained for a class of difference equations satisfying certain symmetry and monotonicity conditions. In particular we consider equations of the form

$$y_n = \min\{f(y_{n-k_1}, y_{n-m_1}), f(y_{n-k_2}, y_{n-m_2})\},$$

where $f(u, v) = h(u, v)/v$ for $h$ symmetric in $u$ and $v$, and $f$ satisfies monotonicity conditions. The results are then extended to the form

$$y_n = \min\{f(y_{n-k_1}, y_{n-m_1}), f(y_{n-k_2}, y_{n-m_2}), \dots, f(y_{n-k_L}, y_{n-m_L})\}.$$

## Extension to $y_n = \min\{f(y_{n-k_1}, y_{n-m_1}), \dots, f(y_{n-k_L}, y_{n-m_L})\}$

**Theorem 3** *Suppose that $\{y_i\}$ satisfies $y_n = f(y_{n-k}, y_{n-m})$ with $f(u, v) = h(u, v)/v$ for some function $h$ where*
*(a) $h \in C((0, \infty)^2, (a, \infty))$ is symmetric in $u$ and $v$ and increasing in $u$,*
*(b) the function $f$ is decreasing in $v$,*
*(c) for all $v > a$ there exist $C_v$ and $D_v$ such that*

$$lim_{u\to a^+} f(u, v)/u = C_v > 1 \text{ and } lim_{u\to\infty} f(u, v)/u = D_v < 1 \tag{6}$$

*Then $\{y_i\}$ is periodic with not necessarily prime period $2$.*

**Theorem 4** *Set $K = \{k_1, k_2, \dots, k_L\}$, $M = \{m_1, m_2, \dots m_L\}$ and $\rho = gcd\{k | k \in K\}$, and suppose the following hold.*

- *The function $f$ satisfies the requirements of the theorem above.*
- *There exists a $T \geq 0$, such that for each $m \in M$, there exists an $m^* \in M$ satisfying*

$$m + m^* = T \mod \rho$$

- *There exists a $Q \geq 0$ and an $m \in M$ such that*

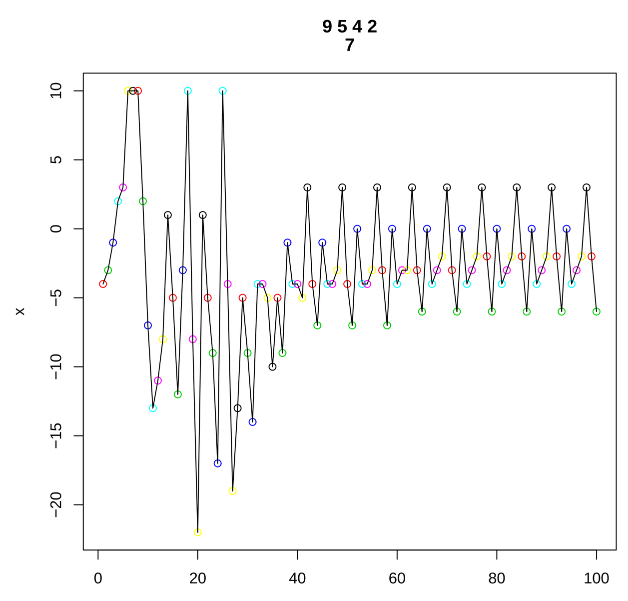
$$m + QT = 0 \mod \rho.$$

*Then every positive solution of*

$$y_n = \min\{f(y_{n-k_1}, y_{n-m_1}), \dots, f(y_{n-k_L}, y_{n-m_L})\}$$

*converges to a unique equilibrium.*

## Tables and Figures



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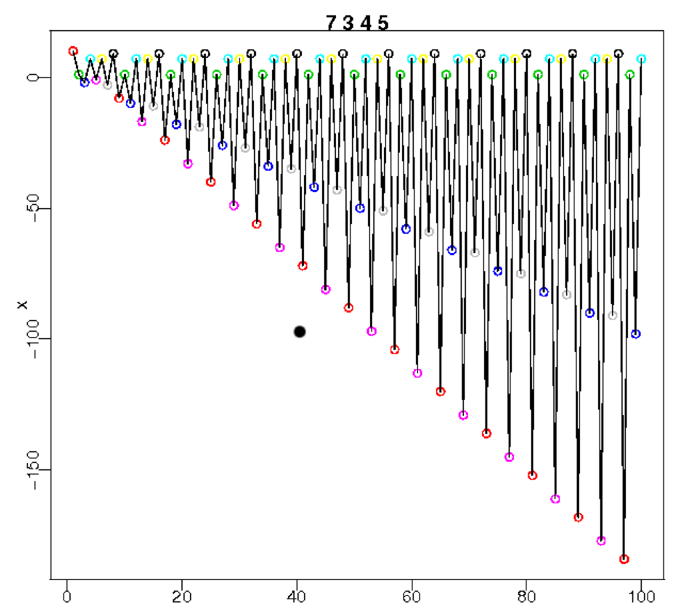
|---|---|---|---|---|---|---|---|---|---|---|
| 13 | 30 | 15 | 21 | 3 | 13 | 9 | 20 | 25 | 27 | −14 |
| −10 | 10 | 6 | −17 | −23 | 1 | −14 | −42 | −50 | 15 | −14 |
| −52 | −56 | 6 | 9 | −53 | −42 | −14 | 20 | −68 | −28 | −14 |
| 0 | −74 | −37 | 9 | −23 | −60 | −57 | 20 | −11 | −46 | −57 |
| 0 | 20 | −55 | −34 | −23 | 23 | −75 | −23 | −11 | −12 | −75 |
| −43 | 20 | −31 | −52 | −66 | 23 | −8 | −41 | −54 | −12 | 35 |
| −61 | −23 | −31 | 19 | −84 | −15 | −8 | −23 | −72 | −50 | 35 |
| −43 | −41 | −69 | 19 | −28 | −33 | −46 | −23 | 22 | −68 | −3 |
| −43 | 20 | −87 | 19 | −22 | −21 | −64 | −23 | 42 | −64 | −21 |
| −43 | 20 | −83 | 1 | −22 | −21 | −60 | −41 | 42 | −64 | −17 |
| −61 | 20 | −65 | 1 | −40 | −21 | −24 | −41 | 24 | −64 | 37 |
| −61 | 20 | −65 | 1 | −40 | −21 | −24 | −41 | 24 | −64 | 37 |



- K. S. Berenhaut, R. T. Guy, "Symmetric Functions and Difference Equations with Asymptotically Period-Two Solutions," in press. (*International Journal of Difference Equations*)
- K. S. Berenhaut, R. T. Guy and C. L. Barrett, "Globally asymptotic behavior for minimum difference equations," in preparation, to be submitted to the *Journal of Difference Equations and Applications.*