SOLUTIONS OF THE CUBIC FERMAT EQUATION IN QUADRATIC FIELDS

MARVIN JONES AND JEREMY ROUSE

Abstract. We give necessary and sufficient conditions on a squarefree integer d for there to be non-trivial solutions to $x^3 + y^3 = z^3$ in $\mathbb{Q}(\sqrt{d})$, conditional on the Birch and Swinnerton-Dyer conjecture. These conditions are similar to those obtained by J. Tunnell in his solution to the congruent number problem.

1. Introduction and Statement of Results

The enigmatic claim of Fermat that the equation

$$x^n + y^n = z^n$$

has only the trivial solutions (those with at least one of x, y and z zero) in integers when $n \geq 3$ has to a large extent shaped the development of number theory over the course of the last three hundred years. These developments culminated in the theory used by Andrew Wiles in [28] to finally justify Fermat's claim.

In light of Fermat's claim and Wiles's proof, it is natural to ask the following question: for which fields K does the equation $x^n + y^n = z^n$ have a non-trivial solution in K? Two notable results on this question are the following. In [16], it is shown that the equation $x^n + y^n = z^n$ has no non-trivial solutions in $\mathbb{Q}(\sqrt{2})$ provided $n \geq 4$. Their proof uses similar ingredients to Wiles's work.

In [10], Debarre and Klassen use Faltings's work on the rational points on subvarieties of abelian varieties to prove that for $n \geq 3$ and $n \neq 6$, the equation $x^n + y^n = z^n$ has only finitely many solutions $(x : y : z)$ where the variables belong to any number field K with $[K : \mathbb{Q}] \leq n - 2$. Indeed, the work of Aigner shows that when $n = 4$ the only non-trivial solution to $x^n + y^n = z^n$ with x, y and z in any quadratic field is

$$\left(\frac{1+\sqrt{-7}}{2}\right)^4 + \left(\frac{1-\sqrt{-7}}{2}\right)^4 = 1^4,$$

and when $n = 6$ or $n = 9$, there are no non-trivial solutions in quadratic fields.

We now turn to the problem of solutions to $x^3 + y^3 = z^3$ in quadratic fields $\mathbb{Q}(\sqrt{d})$. For some choices of d there are solutions, such as

$$(18 + 17\sqrt{2})^3 + (18 - 17\sqrt{2})^3 = 42^3$$

2010 *Mathematics Subject Classification.* Primary 11G05; Secondary 11D41, 11G40, 11F37.

for $d = 2$, while for other choices (such as $d = 3$) there are no non-trivial solutions. In 1913, Fueter [11] showed that if $d < 0$ and $d \equiv 2 \pmod 3$, then there are no solutions if 3 does not divide the class number of $\mathbb{Q}(\sqrt{d})$. Fueter also proved in [12] that there is a non-trivial solution to $x^3 + y^3 = z^3$ in $\mathbb{Q}(\sqrt{d})$ if and only if there is one in $\mathbb{Q}(\sqrt{-3d})$.

In 1915, Burnside [8] showed that every solution to $x^3+y^3 = z^3$ in a quadratic field takes the form

$$\begin{aligned} x &= -3 + \sqrt{-3(1+4k^3)}, \\ y &= -3 - \sqrt{-3(1+4k^3)}, \text{ and} \\ z &= 6k \end{aligned}$$

up to scaling. Here k is any rational number not equal to 0 or -1. This, however, does not answer the question of whether or not there are solutions in $\mathbb{Q}(\sqrt{d})$ for given d since it is not clear whether

$$dy^2 = -3(1+4k^3)$$

has a solution with k and y both rational.

In a series of papers [1], [2], [3], [4], Aigner considered this problem (see [21], Chapter XIII, Section 10 for a discussion in English). He showed that there are no solutions in $\mathbb{Q}(\sqrt{-3d})$ if $d > 0$, $d \equiv 1 \pmod 3$, and 3 does not divide the class number of $\mathbb{Q}(\sqrt{-3d})$. He also developed general criteria to rule out the existence of a solution. In particular, there are "obstructing integers" k with the property that there are no solutions in $\mathbb{Q}(\sqrt{\pm d})$ if $d = kR$, where R is a product of primes congruent to 1 (mod 3) for which 2 is a cubic non-residue.

The goal of the present paper is to give a complete classification of the fields $\mathbb{Q}(\sqrt{d})$ in which $x^3 + y^3 = z^3$ has a solution. Our main result is the following.

Theorem 1. *Assume the Birch and Swinnerton-Dyer conjecture (see Section 2 for the statement and background). If $d > 0$ is squarefree with $\gcd(d,3) = 1$, then there is a non-trivial solution to $x^3 + y^3 = z^3$ in $\mathbb{Q}(\sqrt{d})$ if and only if*

$$\begin{aligned} &\#\{(x,y,z) \in \mathbb{Z}^3 : x^2 + y^2 + 7z^2 + xz = d\} \\ &= \#\{(x,y,z) \in \mathbb{Z}^3 : x^2 + 2y^2 + 4z^2 + xy + yz = d\}. \end{aligned}$$

If $d > 0$ is squarefree with $3|d$, then there is a non-trivial solution to $x^3 + y^3 = z^3$ in $\mathbb{Q}(\sqrt{d})$ if and only if

$$\begin{aligned} &\#\{(x,y,z) \in \mathbb{Z}^3 : x^2 + 3y^2 + 27z^2 = d/3\} \\ &= \#\{(x,y,z) \in \mathbb{Z}^3 : 3x^2 + 4y^2 + 7z^2 - 2yz = d/3\}. \end{aligned}$$

Moreover, there are non-trivial solutions in $\mathbb{Q}(\sqrt{d})$ if and only if there are non-trivial solutions in $\mathbb{Q}(\sqrt{-3d})$.

Remark. *Only one direction of our result is conditional on the Birch and Swinnerton-Dyer conjecture. As mentioned in Section 2, it is known that if $E/\mathbb{Q}$ is an elliptic curve, $L(E,1) \neq 0$ implies that $E(\mathbb{Q})$ is finite. As a consequence, if the number of representations of d (respectively $d/3$) by the two different quadratic forms are different, then there are no solutions in $\mathbb{Q}(\sqrt{d})$.*

Our method is similar to that used by Tunnell [26] in his solution to the congruent number problem. The congruent number problem is to determine, given a positive integer n, whether there is a right triangle with rational side lengths and area n. It can be shown that n is a congruent number if and only if the elliptic curve $E_n : y^2 = x^3 - n^2x$ has positive rank. The Birch and Swinnerton-Dyer states that E_n has positive rank if and only if $L(E_n, 1) \neq 0$, and Waldspurger's theorem (roughly speaking) states that

$$f(z) = \sum_{n=1}^{\infty} n^{1/4}\sqrt{L(E_{-n},1)}q^n, \quad q = e^{2\pi i z}$$

is a weight 3/2 modular form. Tunnell computes this modular form explicitly as a difference of two weight 3/2 theta series and proves that (in the case that n is odd), n is congruent if and only if n has the same number of representations in the form $x^2 + 4y^2 + 8z^2$ with z even as it does with z odd. Tunnell's work was used in [14] to determine precisely which integers $n \leq 10^{12}$ are congruent (again assuming the Birch and Swinnerton-Dyer conjecture).

Remark. *In [20], Soma Purkait computes two (different) weight 3/2 modular forms whose coefficients interpolate the central critical L-values of twists of $x^3 + y^3 = z^3$ (see Proposition 8.7). Purkait expresses the first as a linear combination of 7 theta series, but does not express the second in terms of theta series.*

An outline of the paper is as follows. In Section 2 we will discuss the Birch and Swinnerton-Dyer conjecture. In Section 3 we will develop the necessary background. This will be used in Section 4 to prove Theorem 1.

Acknowledgements. *This work represents the master's thesis of the first author which was completed at Wake Forest University. The authors used Magma [5] version 2.17 for computations in spaces of modular forms of integer and half-integer weights. The authors also thank the anonymous referee for many helpful suggestions which have substantially improved the paper.*

2. Elliptic Curves and the Birch and Swinnerton-Dyer Conjecture

The smooth, projective curve $C : x^3 + y^3 = z^3$ is an elliptic curve. Specifically, if $X = \frac{12z}{y+x}$ and $Y = \frac{36(y-x)}{y+x}$, then

$$E_1 : Y^2 = X^3 - 432.$$

From Euler's proof of the $n = 3$ case of Fermat's last theorem, it follows that the only rational points on $x^3 + y^3 = z^3$ are $(1 : 0 : 1)$, $(0 : 1 : 1)$, and $(1 : -1 : 0)$. These correspond to the three-torsion points $(12, -36)$, $(12, 36)$, and the point at infinity on E_1.

Suppose that $K = \mathbb{Q}(\sqrt{d})$ is a quadratic field and $\sigma : K \to K$ is the automorphism given by $\sigma(a + b\sqrt{d}) = a - b\sqrt{d}$ with $a, b \in \mathbb{Q}$. If $P = (x, y) \in E_1(K)$, define $\sigma(P) = (\sigma(x), \sigma(y)) \in E_1(K)$. Then, $Q = P - \sigma(P) \in E_1(K)$ and $\sigma(Q) = -Q$. Since the inverse of $(x, y) \in E_1(K)$ is $(x, -y)$, it follows that $P - \sigma(P) = (a, b\sqrt{d})$ for $a, b \in \mathbb{Q}$. Thus, (a, b) is a rational point on the quadratic twist E_d of E, given by

$$E_d : dY^2 = X^3 - 432.$$

Lemma 2. *The point (a, b) on $E_d(\mathbb{Q})$ is in the torsion subgroup of $E_d(\mathbb{Q})$ if and only if the corresponding solution to $x^3 + y^3 = z^3$ is trivial.*

This lemma will be proven in Section 4. Thus, there is a non-trivial solution in $\mathbb{Q}(\sqrt{d})$ if and only if $E_d(\mathbb{Q})$ has positive rank.

If $E/\mathbb{Q}$ is an elliptic curve, let

$$L(E, s) = \sum_{n=1}^{\infty} \frac{a_n(E)}{n^s}$$

be its L-function (see [24], Appendix C, Section 16 for the precise definition). It is known (see [6]) that $L(E, s) = L(f, s)$ for some weight 2 modular form $f \in S_2(\Gamma_0(N))$, where N is the conductor of E. It follows from this that $L(E, s)$ has an analytic continuation and functional equation of the form

$$\Lambda(E, s) = (2\pi)^{-s} N^{s/2} \Gamma(s) L(E, s)$$

and $\Lambda(E, s) = w_E \Lambda(E, 2 - s)$, where $w_E = \pm 1$ is the root number of E. Note that if $w_E = -1$, then $L(E, 1) = 0$. The weak Birch and Swinnerton-Dyer conjecture predicts that

$$\operatorname{ord}_{s=1} L(E, s) = \operatorname{rank}(E(\mathbb{Q})).$$

The strong form predicts that

$$\lim_{s \to 1} \frac{L(E, s)}{(s - 1)^r} = \frac{\Omega(E) R(E/\mathbb{Q}) \prod_p c_p \# Ш(E/\mathbb{Q})}{(\# E_{\text{tors}})^2}.$$

Here, $\Omega(E)$ is the real period of E times the number of connected components of $E(\mathbb{R})$, $R(E/\mathbb{Q})$ is the elliptic regulator, the c_p are the Tamagawa numbers, and $Ш(E/\mathbb{Q})$ is the Shafarevich-Tate group.

Much is known about the Birch and Swinnerton-Dyer in the case when $\operatorname{ord}_{s=1} L(E, s)$ is 0 or 1. See for example [9], [13], [17], and [22]. The best known result currently is the following.

Theorem 3 (Gross-Zagier, Kolyvagin, et al.)**.** *Suppose that $E/\mathbb{Q}$ is an elliptic curve and $\operatorname{ord}_{s=1} L(E, s) = 0$ or 1. Then, $\operatorname{ord}_{s=1} L(E, s) = \operatorname{rank}(E(\mathbb{Q}))$.*

The work of Bump-Friedberg-Hoffstein [7] or Murty-Murty [18] is necessary to remove a condition imposed in the work of Gross-Zagier and Kolyvagin.

3. Preliminaries

If d is an integer, let χ_d denote the unique primitive Dirichlet character with the property that

$$\chi_d(p) = \left(\frac{d}{p}\right)$$

for all odd primes p. This character will be denoted by $\chi_d(n) = \left(\frac{d}{n}\right)$, even when n is not prime.

If λ is a positive integer, let $M_{2\lambda}(\Gamma_0(N), \chi)$ denote the $\mathbb{C}$-vector space of modular forms of weight 2λ for $\Gamma_0(N)$ with character χ, and $S_{2\lambda}(\Gamma_0(N), \chi)$ denote the subspace of cusp forms. Similarly, if λ is a positive integer, let $M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$ denote the vector space of modular forms of weight $\lambda + \frac{1}{2}$ on $\Gamma_0(4N)$ with character χ and $S_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$ denote the subspace of cusp forms. We will frequently use the following theorem of Sturm [25] to prove that two modular forms are equal.

Theorem 4. *Suppose that $f(z) \in M_r(\Gamma_0(N), \chi)$ is a modular form of integer or half-integer weight with $f(z) = \sum_{n=0}^{\infty} a(n)q^n$. If $a(n) = 0$ for $n \le \frac{r}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$, then $f(z) = 0$.*

We denote by T_p the usual index p Hecke operator on $M_{2\lambda}(\Gamma_0(N), \chi)$, and by T_{p^2} the usual index p^2 Hecke operator on $M_{\lambda+1/2}(\Gamma_0(4N), \chi)$. If d is a positive integer, we define the operator $V(d)$ by

$$\left(\sum a(n)q^n\right) |V(d) = \sum a(n)q^{dn}.$$

Next, we recall the Shimura correspondence.

Theorem 5 ([23])**.** *Suppose that $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{\lambda+1/2}(\Gamma_0(4N), \chi)$. For each squarefree integer t, let*

$$\mathcal{S}_t(f(z)) = \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi(d) \left(\frac{(-1)^\lambda t}{d}\right) d^{\lambda-1} a(t(n/d)^2) \right) q^n.$$

Then, $\mathcal{S}_t(f(z)) \in M_{2\lambda}(\Gamma_0(2N), \chi^2)$.

One can show using the definition that if p is a prime and $p \nmid 4tN$, then

$$\mathcal{S}_t(f|T_{p^2}) = \mathcal{S}_t(f)|T_p,$$

that is, the Shimura correspondence commutes with the Hecke action.

In [27], Waldspurger relates the Fourier coefficients of a half-integer weight Hecke eigenform f with the central critical L-values of the twists of the corresponding integer weight modular form g with the same Hecke eigenvalues. Recall that if

$$F(z) = \sum_{n=1}^{\infty} A(n) q^n,$$

then $(F \otimes \chi)(z) = \sum_{n=1}^{\infty} A(n)\chi(n) q^n$.

Theorem 6 ([27], Corollaire 2, p. 379)**.** *Suppose that $f \in S_{\lambda+1/2}(\Gamma_0(4N), \chi)$ is a half-integer weight modular form and $f|T_{p^2} = \lambda(p) f$ for all $p \nmid 4N$. Denote the Fourier expansion of $f(z)$ by*

$$f(z) = \sum_{n=1}^{\infty} a(n) q^n.$$

If $F(z) \in S_{2\lambda}(\Gamma_0(2N), \chi^2)$ is an integer weight modular form with $F(z)|T_p = \lambda(p) g$ for all $p \nmid 4N$ and n_1 and n_2 are two squarefree positive integers with $n_1/n_2 \in \left(\mathbb{Q}_p^\times\right)^2$ for all $p|N$, then

$$a(n_1)^2 L(F \otimes \chi^{-1}\chi_{n_2 \cdot (-1)^\lambda}, \lambda)\chi(n_2/n_1) n_2^{\lambda-1/2} = a(n_2)^2 L(F \otimes \chi^{-1}\chi_{n_1 \cdot (-1)^\lambda}, \lambda) n_1^{\lambda-1/2}.$$

Our goal is to construct two modular forms $f_1(z) \in S_{3/2}(\Gamma_0(108))$ and $f_2(z) \in S_{3/2}(\Gamma_0(108), \chi_3)$ whose Fourier coefficients encode the L-values of twists of $E_1 : y^2 = x^3 - 432$. This is the (unique up to isogeny) elliptic curve of conductor 27. By the modularity of elliptic curves it therefore corresponds to the unique normalized weight 2 cusp form of level 27

$$F(z) = q \prod_{n=1}^{\infty} (1 - q^{3n})^2 (1 - q^{9n})^2 \in S_2(\Gamma_0(27)).$$

As in [26], we will express f_1 and f_2 as linear combinations of ternary theta functions. The next result recalls the modularity of the theta series of positive-definite quadratic forms.

Theorem 7 (Theorem 10.9 of [15])**.** *Let A be an $r \times r$ positive-definite symmetric matrix with integer entries and even diagonal entries. Let $Q(\vec{x}) = \frac{1}{2}\vec{x}^T A \vec{x}$, and let*

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) q^n$$

be the generating function for the number of representations of n by Q. Then,

$$\theta_Q(z) \in M_{r/2}(\Gamma_0(N), \chi_{\det(2A)}),$$

where N is the smallest positive integer so that NA^{-1} has integer entries and even diagonal entries.

Finally, we require some facts about the root numbers of the curves E_d. If $F(z) \in S_2(\Gamma_0(N))$ is the modular form corresponding to E, let $F(z)|W(N) = N^{-1}z^{-2}F\left(-\frac{1}{Nz}\right)$. Then $F(z)|W(N) = -w_E F(z)$ (see for example Theorem 7.2 of [15]). Theorem 7.5 of [15] states that if ψ is a quadratic Dirichlet character with conductor r and $\gcd(r, N) = 1$, then $F \otimes \psi \in S_2(\Gamma_0(Nr^2))$ and

$$(F \otimes \psi)|W(Nr^2) = (\psi(N)\tau(\psi)^2/r)F|W(N)$$

where $\tau(\psi) = \sum_{m=1}^{r} \psi(m)e^{2\pi i m/r}$ is the usual Gauss sum.

Suppose d is an integer so that $|d|$ is the conductor of χ_d and $F(z) \in S_2(\Gamma_0(27))$ is the modular form corresponding to E_1. Then $F \otimes \chi_d$ is the modular form corresponding to E_d. Using the result from the previous paragraph and the equality $\tau(\chi_d)^2 = |d|\chi_d(-1)$, we get

$$w_{E_d} = w_{E_1}\chi_d(27)\chi_d(-1) = \chi_d(-27).$$

provided $\gcd(d, 3) = 1$.

4. Proofs

In this section, we prove Lemma 2 and Theorem 1.

Before we prove Lemma 2, we will first need to determine the order of the torsion subgroup of $E_d(\mathbb{Q})$. Since $x^3 - 432d^3$ has no rational roots, there are no elements of order two in $E_d(\mathbb{Q})$ and so $|E_d(\mathbb{Q})_{\text{tors}}|$ is odd. We will now show $q \nmid E_d(\mathbb{Q})_{\text{tors}}$ for primes $q > 3$.

If p is prime with $p \equiv 2 \pmod 3$, then we have that the map $x \to x^3 \in \mathbb{F}_p$ is a bijection. From this it follows that $\sum_{x=0}^{p-1}\left(\frac{f(x)}{p}\right) = 0$. Thus we have $\#E(\mathbb{F}_p) = p + 1$. Suppose that $|E_d(\mathbb{Q})_{\text{tors}}| = N$ for N odd.

If we suppose that a prime $q > 3$ divides N then we can find an integer x that is relatively prime to $3q$ so that $x \equiv 2 \pmod 3$ and $x \equiv 1 \pmod q$. By Dirichlet's Theorem, we have an infinite number of primes contained in the arithmetic progression $3nq + x$ for $n \in \mathbb{N}$. If we take p to be a sufficiently large prime in this progression, then the reduction of $E_d(\mathbb{Q})_{\text{tors}} \subseteq E(\mathbb{F}_p)$ has order N. So, now we have that q divides $|E_d(\mathbb{F}_p)| = p+1 \equiv x+1 \equiv 2 \pmod q$. This is a contradiction. Hence the only prime that can divide N is 3. We can follow a similar argument to show that 9 does not divide N. This means that the torsion subgroup of $E_d(\mathbb{Q})$ is either $\mathbb{Z}/3\mathbb{Z}$ or trivial.

Furthermore, if $E_d(\mathbb{Q})$ contains a point of order 3 then the x-coordinate of the point must be a root of the three-division polynomial $\phi_3(x) = 3x^4 - 12(432)d^3x$. The only real roots to $\phi_3(x)$ are $x = 0$ and $x = 12d$. For $x = 0$, then we have $y = \pm 108$ and $d = -3$. If $x = 12d$, then we find that $y = 1296d^3$ and that $d = 1$. Thus we conclude that $E_d(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/3\mathbb{Z}$ if $d = 1$ or $d = -3$, and $E_d(\mathbb{Q})_{\text{tors}}$ is trivial otherwise.

Proof of Lemma 2. ($\Rightarrow$) Let $(x, y) \in E_d(\mathbb{Q})$ so that (x, y) is not in $E_d(\mathbb{Q})_{\text{tors}}$. By doing some arithmetic we get that $(x, y\sqrt{d}) \in E(K)$. In Section 2, we defined a map from $C(K) \to E(K)$. The inverse of this map sends

$$(x, y\sqrt{d}) \to \left(\frac{\frac{12}{x} + \frac{y\sqrt{d}}{3x}}{2}, \frac{\frac{12}{x} - \frac{y\sqrt{d}}{3x}}{2} \right) \in C(K).$$

If we suppose that this is a trivial solution to C, then either the x-coordinate or y-coordinate is zero. Hence $y = \pm\frac{36\sqrt{d}}{d}$.

If $d = 1$, then we have $y = -36$ and $x = 12$. From Section 2, we know that $(12, -36)$ corresponds to $(1 : 0 : 1)$ which is a trivial solution to C. Hence the point (x, y) does not satisfy the hypothesis for $d = 1$. Now for $d \neq 1$, we have $y \notin \mathbb{Q}$. This contradicts the hypothesis for $d \neq 1$. Hence the solution we have is non-trivial.

($\Leftarrow$) Let (x, y, z) be a non-trivial solution to $x^3 + y^3 = z^3$ in K. Note that for $d = 1$ or -3 Euler showed that there are only trivial solutions and thus this direction is vacuously true for these two cases.

For $d \neq 1$ and -3, from Section 2 we showed that $(x, y, z) \to (X, Y) = P \in E(K)$. Also from section 2, if $P - \sigma(P) = (a, b\sqrt{d})$ then $(a, b) \in E_d(\mathbb{Q})$. Since $d \neq 1$ and -3, then the torsion subgroup of $E_d(\mathbb{Q})$ is trivial. Thus $(a, b) \notin E_d(\mathbb{Q})_{\text{tors}}$. ☐

Recall from Section 3 that the elliptic curve E_1 corresponds to the modular form $F(z) = q\prod_{n=1}^{\infty}(1 - q^{3n})^2(1 - q^{9n})^2 \in S_2(\Gamma_0(27))$.

Remark. *For convenience we will think of $F(z)$ as a Fourier series with coefficients $\lambda(n)$ for $n \in \mathbb{N}$. Note that if $\lambda(n) \neq 0$ then $n \equiv 1 \pmod 3$. So we can write $\lambda(n) = \lambda(n)(\frac{n}{3})$ for $n \in \mathbb{N}$. Hence $F \otimes \chi_{-3d} = F \otimes \chi_d$. We can now conclude that $L(E_d, s) = L(E_{-3d}, s)$.*

Proof of Theorem 1. We will begin with an outline. Lemma 2 implies that there are non-trivial solutions to $x^3 + y^3 = z^3$ in $\mathbb{Q}(\sqrt{d})$ if and only if $E_d(\mathbb{Q})$ has positive rank. The Birch and Swinnerton-Dyer conjecture states that this occurs if and only if $L(E_d, 1) = 0$. Waldspurger's theorem relates the dth coefficient of a form (expressible as the difference of two ternary theta series) in $S_{3/2}(\Gamma_0(108), \chi_1)$ to $L(E_{-d}, 1)$ and the dth coefficient of a form in $S_{3/2}(\Gamma_0(108), \chi_3)$ to $L(E_{-3d}, 1)$. This gives us control of solutions to $x^3 + y^3 = z^3$ in imaginary quadratic fields. Since $L(E_d, 1) = L(E_{-3d}, 1)$, the case of real quadratic fields is determined by imaginary quadratic fields.

We start by finding a formula for $L(E_d, 1)$ where $d < 0$ and $d \equiv 2 \pmod 3$. To meet the hypotheses of Waldspurger's theorem, we need a Hecke eigenform in $S_{3/2}(\Gamma_0(108), \chi_1)$ with the same Hecke eigenvalues as $F(z)$ for $p > 3$. Note that

$\dim S_{3/2}(\Gamma_0(108), \chi_1) = 5$. Moreover, we have the following basis of $S_{3/2}(\Gamma_0(108), \chi_1)$:

$$
\begin{aligned}
g_1(z) &= q - q^{10} - q^{16} - q^{19} - q^{22} + 2q^{28} + \cdots, \\
g_2(z) &= q^2 - q^5 + q^8 - q^{11} + q^{14} - 2q^{17} - q^{20} + \cdots, \\
g_3(z) &= q^3 - 2q^{12} + \ldots, \\
g_4(z) &= q^4 - q^{10} + q^{13} - q^{16} - q^{19} - q^{22} - q^{25} + q^{28} + \cdots, \text{ and} \\
g_5(z) &= q^7 - q^{10} + q^{13} - q^{16} - q^{22} - q^{25} + \cdots.
\end{aligned}
$$

To prove that a linear combination of the $g_i(z)$ is an eigenform, we will use properties of the Shimura correspondence $\mathcal{S}_t$ for $t = 1$, 2 and 3. By Theorem 4 we have:

$$
\begin{aligned}
\mathcal{S}_1(g_1(z) + g_4(z)) &= F(z) + F(z)|V(2), \\
\mathcal{S}_2(g_1(z) + g_4(z)) &= 0, \\
\mathcal{S}_3(g_1(z) + g_4(z)) &= 0, \\
\mathcal{S}_1(g_1(z) + g_5(z)) &= F(z), \\
\mathcal{S}_2(g_1(z) + g_5(z)) &= 0, \text{ and} \\
\mathcal{S}_3(g_1(z) + g_5(z)) &= 0.
\end{aligned}
$$

Note that $F(z)$ and $F(z)|V(2)$ are both eigenforms of T_p for primes $p > 3$ with the same Hecke eigenvalues $\lambda(p)$. It follows that $\mathcal{S}_t((g_1(z)+g_4(z))|T_{p^2}) = \mathcal{S}_t((g_1(z)+g_4(z))|T_p$ for all primes $p > 3$. It follows that $(g_1(z)+g_4(z))|T_{p^2} - \lambda(p)(g_1(z)+g_4(z))$ is in $\ker(\mathcal{S}_1)$, $\ker(\mathcal{S}_2)$, and $\ker(\mathcal{S}_3)$. Furthermore since $\ker(\mathcal{S}_1) \cap \ker(\mathcal{S}_2) \cap \ker(\mathcal{S}_3) = 0$, it follows that $g_1(z)+g_4(z)$ is a half-integer weight Hecke eigenform. A similar argument proves the same for $g_1(z) + g_5(z)$.

We will now take the quadratic forms $Q_1(x, y, z) = x^2+3y^2+27z^2$ and $Q_2(x, y, z) = 3x^2 + 4y^2 - 2yz + 7z^2$. Their theta-series θ_{Q_1} and θ_{Q_2} are in $M_{3/2}(\Gamma_0(108), \chi_1)$. Also by Theorem 4, we have

$$\theta_{Q_1}(z) - \theta_{Q_2}(z) = -2(g_1(z) + g_5(z)) + 4(g_1(z) + g_4(z)).$$

Furthermore, since $g_1(z) + g_4(z)$ and $g_1(z) + g_5(z)$ are both Hecke eigenforms with the same eigenvalues then $\theta_{Q_1}(z) - \theta_{Q_2}(z)$ is a Hecke eigenform as well.

Let $a(n)$ denote the nth coefficient of $\theta_{Q_1}(z) - \theta_{Q_2}(z)$. By Theorem 6, we have

$$L(E_{-n_2}, 1) = \sqrt{\frac{n_2}{n_1}} \left(\frac{a(n_2)}{a(n_1)} \right)^2 L(E_{-n_1}, 1)$$

provided n_1 and n_2 are squarefree and n_1/n_2 is a square in $\mathbb{Q}_2$ and in $\mathbb{Q}_3$. For the time being we will consider the case that $n_1 \equiv n_2 \equiv 1 \pmod 3$. We have 8 cases (corresponding to the 8 square classes in $\mathbb{Q}_2^\times$).

n_2	n_1	$a(n_1)$	$L(E_{-n_1}, 1)$
$n_2 \equiv 1 \pmod{24}$	1	2	$1.52995\ldots$
$n_2 \equiv 34 \pmod{48}$	34	4	$1.04953\ldots$
$n_2 \equiv 19 \pmod{24}$	19	-6	$0.70199\ldots$
$n_2 \equiv 13 \pmod{24}$	13	2	$0.42434\ldots$
$n_2 \equiv 22 \pmod{48}$	22	-4	$1.30474\ldots$
$n_2 \equiv 7 \pmod{24}$	7	-2	$1.15653\ldots$
$n_2 \equiv 10 \pmod{36}$	10	-4	$1.93525\ldots$
$n_2 \equiv 46 \pmod{48}$	46	4	$0.90231\ldots$

Thus, when $d < 0$ with $d \equiv 2 \pmod{3}$ we have that $L(E_d, 1) = 0$ if and only if $a(-d) = 0$. Since $L(E_{-3d}, 1) = L(E_d, 1)$, we have a formula for $L(E_{-3d}, 1)$ when $-3d > 0$ and $-3d \equiv 3 \pmod{9}$.

We will now examine the case when $d < 0$ and $d \equiv 6 \pmod{9}$. Note that $\dim S_{3/2}(\Gamma_0(108), \chi_3) = 5$, and we have the basis:

$$\begin{aligned}
h_1(z) &= q - 2q^{13} - q^{25} - 2q^{28} + \cdots, \\
h_2(z) &= q^2 + q^5 - q^8 - q^{11} - q^{14} - q^{20} - 2q^{23} + 2q^{26} + \cdots, \\
h_3(z) &= q^4 - q^{13} - 2q^{16} + 2q^{25} - q^{28} + \cdots, \\
h_4(z) &= q^7 - q^{13} - q^{19} + \cdots, \text{ and} \\
h_5(z) &= q^{10} - q^{16} - q^{19} - q^{22} + q^{25} + \cdots.
\end{aligned}$$

By Theorem 4, we have

$$\begin{aligned}
\mathcal{S}_1(h_1(z) - h_4(z) + 2h_5(z)) &= F(z), \\
\mathcal{S}_2(h_1(z) - h_4(z) + 2h_5(z)) &= 0, \\
\mathcal{S}_3(h_1(z) - h_4(z) + 2h_5(z)) &= 0, \\
\mathcal{S}_1(h_1(z) - 4h_3(z) - 5h_4(z) + 10h_5(z)) &= F(z) + 4F(z)|V(2), \\
\mathcal{S}_2(h_1(z) - 4h_3(z) - 5h_4(z) + 10h_5(z)) &= 0, \text{ and} \\
\mathcal{S}_3(h_1(z) - 4h_3(z) - 5h_4(z) + 10h_5(z)) &= 0.
\end{aligned}$$

From a similar argument as the previous case, we get that $h_1(z) - h_4(z) + 2h_5(z)$ and $h_1(z) - 4h_3(z) - 5h_4(z) + 10h_5(z)$ are Hecke eigenforms for T_{p^2} for primes $p > 3$ with the same eigenvalues as $F(z)$. We will now take the quadratic forms $Q_3(x, y, z) = x^2 + y^2 + 7z^2 + xz$ and $Q_4(x, y, z) = x^2 + 2y^2 + 4z^2 + xy + yz$. We have that $\theta_{Q_3}, \theta_{Q_4} \in M_{3/2}(\Gamma_0(108), \chi_3)$. By Theorem 4, $\theta_{Q_3} - \theta_{Q_4} = 2h_1(z) - 4h_3(z) - 6h_4(z) + 12h_5(z)$. This is the sum of $h_1(z) - h_4(z) + 2h_5(z)$ and $h_1(z) - 4h_3(z) - 5h_4(z) + 10h_5(z)$, which are Hecke eigenforms with the same eigenvalues. It follows that $\theta_{Q_3} - \theta_{Q_4}$ is a Hecke eigenform. Let $b(n)$ denote the nth coefficient of $\theta_{Q_3} - \theta_{Q_4}$.

Hence by Theorem 6, we have

$$L(E_{-3n_2},1)=\sqrt{\frac{n_2}{n_1}}\left(\frac{b(n_2)}{b(n_1)}\right)^2 L(E_{-3n_1},1),$$

provided n_1 and n_2 are squarefree and n_1/n_2 is a 2-adic square and a 3-adic square. Again we consider the cases that $n_1 \equiv n_2 \equiv 1 \pmod 3$.

n_2	n_1	$b(n_1)$	$L(E_{-3n_1},1)$
$n_2 \equiv 1 \pmod{24}$	1	2	$0.58887\ldots$
$n_2 \equiv 34 \pmod{48}$	34	12	$1.81785\ldots$
$n_2 \equiv 19 \pmod{24}$	19	-6	$0.60794\ldots$
$n_2 \equiv 13 \pmod{24}$	13	6	$1.46993\ldots$
$n_2 \equiv 22 \pmod{48}$	22	-12	$2.25989\ldots$
$n_2 \equiv 7 \pmod{24}$	7	-6	$1.00159\ldots$
$n_2 \equiv 10 \pmod{36}$	10	12	$3.35196\ldots$
$n_2 \equiv 46 \pmod{48}$	46	-12	$1.56286\ldots$

Therefore if $d<0$ and $d \equiv 6 \pmod 9$, $L(E_d,1)=0$ if and only if $b(-d/3)=0$. Again, we have $L(E_{-d/3},1)=L(E_{-3d},1)=L(E_d,1)$ and so we also handle cases where $d>0$ and $d\equiv 1 \pmod 3$.

In both of the preceding arguments, we have only considered the cases that $n_1 \equiv n_2 \equiv 1 \pmod 3$. In fact, as we will now show, the nth coefficients of $\theta_{Q_1}-\theta_{Q_2}$ and $\theta_{Q_3}-\theta_{Q_4}$ vanish when $n \equiv 2 \pmod 3$. We will show that in the corresponding arithmetic progressions, the root number of E_d is -1 and hence $L(E_d,1)=0$.

Suppose that $d>0$ with $d\equiv 2 \pmod 3$. Recall from the end of Section 3 that $w_{E_d}=\chi_d(-27)$. Hence $w_{E_d}=\chi_d(-27)=\left(\frac{d}{-27}\right)=-1$. Since $L(E_d,s)=L(E_{-3d},s)$, we have $w_{E_d}=-1$ when $d<0$ with $d\equiv 3 \pmod 9$. A simple calculation shows that E_{-1} has conductor 432 and $w_{E_{-1}}=1$. This implies that $w_{E_{-d}}=w_{E_d}$ and so $w_{E_d}=-1$ when $d>0$ with $d\equiv 6 \pmod 9$ and when $d<0$ with $d \equiv 2 \pmod 3$.

Now we will show that when $d>0$ and $d\equiv 6 \pmod 9$, the coefficient of q^d in $\theta_{Q_1}-\theta_{Q_2}$ is zero. This follows from the observation that $x^2+3y^2+27z^2 \equiv 0$ or $1 \pmod 3$. Also $3x^2+4y^2-2yz+7z^2 \equiv (y+2z)^2 \equiv 0$ or $1 \pmod 3$. Hence $r_{Q_1}(d)=r_{Q_2}(d)=0$.

A more involved argument is necessary for the coefficient of q^d in $\theta_{Q_3}-\theta_{Q_4}$. Let ψ be the non-trivial Dirichlet character with modulus 3. Note that $(\theta_{Q_3}(z)-\theta_{Q_4}(z))\otimes\psi \in M_{3/2}(\Gamma_0(108\cdot 3^2),\chi_3\psi^2)$ by Proposition 3.12 in [19]. By Theorem 4, $(\theta_{Q_3}(z)-\theta_{Q_4}(z))\otimes \psi=\theta_{Q_3}(z)-\theta_{Q_4}(z)$. This gives that $b(n)\psi(n)=b(n)$ for all $n\geq 1$ and this implies that $r_{Q_3}(d)-r_{Q_4}(d)=0$ if $d\equiv 2 \pmod 3$.

Hence we have shown that checking the number of solutions of the pair of equations $x^2+y^2+7z^2+xz=d$ and $x^2+2y^2+4z^2+xy+yz=d$, or $x^2+3y^2+27z^2=d/3$

and $3x^2 + 4y^2 + 7z^2 - 2yz = d/3$ is sufficient to determine when there are non-trivial solutions to $x^3 + y^3 = z^3$ in $\mathbb{Q}(\sqrt{d})$. $\square$

References

[1] Alexander Aigner. Ein zweiter Fall der Unmöglichkeit von $x^3+y^3 = z^3$ in quadratischen Körpern mit durch 3 teilbarer Klassenzahl. *Monatsh. Math.*, 56:335–338, 1952.

[2] Alexander Aigner. Weitere Ergebnisse über $x^3 + y^3 = z^3$ in quadratischen Körpern. *Monatsh. Math.*, 56:240–252, 1952.

[3] Alexander Aigner. Unmöglichkeitskernzahlen der kubischen Fermatgleichung mit Primfaktoren der Art $3n + 1$. *J. Reine Angew. Math.*, 195:175–179 (1956), 1955.

[4] Alexander Aigner. Die kubische Fermatgleichung in quadratischen Körpern. *J. Reine Angew. Math.*, 195:3–17 (1955), 1956.

[5] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[6] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor. On the modularity of elliptic curves over $\mathbf{Q}$: wild 3-adic exercises. *J. Amer. Math. Soc.*, 14(4):843–939 (electronic), 2001.

[7] Daniel Bump, Solomon Friedberg, and Jeffrey Hoffstein. Nonvanishing theorems for L-functions of modular forms and their derivatives. *Invent. Math.*, 102(3):543–618, 1990.

[8] William Burnside. On the rational solutions of the equation $x^3 + y^3 + z^3 = 0$ in quadratic fields. *Proc. London Math. Soc.*, 14:1–4, 1915.

[9] John Coates and Andrew Wiles. On the conjecture of Birch and Swinnerton-Dyer. *Invent. Math.*, 39(3):223–251, 1977.

[10] Olivier Debarre and Matthew J. Klassen. Points of low degree on smooth plane curves. *J. Reine Angew. Math.*, 446:81–87, 1994.

[11] Rudolf Fueter. Die diophantische Gleichung $\xi^3 + \eta^3 + \zeta^3 = 0$. *Abh. Akad. Wiss. Heidelberg*, 25, 1913.

[12] Rudolf Fueter. Über kubische diophantische Gleichungen. *Comment. Math. Helv.*, 2(1):69–89, 1930.

[13] Benedict H. Gross and Don B. Zagier. Heegner points and derivatives of L-series. *Invent. Math.*, 84(2):225–320, 1986.

[14] William B. Hart, Gonzalo Tornaría, and Mark Watkins. Congruent number theta coefficients to 10^{12}. In *Algorithmic number theory*, volume 6197 of *Lecture Notes in Comput. Sci.*, pages 186–200. Springer, Berlin, 2010.

[15] Henryk Iwaniec. *Topics in classical automorphic forms*, volume 17 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.

[16] Frazer Jarvis and Paul Meekin. The Fermat equation over $\mathbb{Q}(\sqrt{2})$. *J. Number Theory*, 109(1):182–196, 2004.

[17] Victor A. Kolyvagin. Finiteness of $E(\mathbf{Q})$ and $Ш(E, \mathbf{Q})$ for a subclass of Weil curves. *Izv. Akad. Nauk SSSR Ser. Mat.*, 52(3):522–540, 670–671, 1988.

[18] M. Ram Murty and V. Kumar Murty. Mean values of derivatives of modular L-series. *Ann. of Math. (2)*, 133(3):447–475, 1991.

[19] Ken Ono. *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, volume 102 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2004.

[20] Soma Purkait. Explicit applications of Waldspurger's theorem. Preprint.

[21] Paulo Ribenboim. *13 lectures on Fermat's last theorem.* Springer-Verlag, New York, 1979.

[22] Karl Rubin. Tate-Shafarevich groups and L-functions of elliptic curves with complex multiplication. *Invent. Math.*, 89(3):527–559, 1987.

[23] Goro Shimura. On modular forms of half integral weight. *Ann. of Math. (2)*, 97:440–481, 1973.

[24] Joseph H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1992. Corrected reprint of the 1986 original.

[25] Jacob Sturm. On the congruence of modular forms. In *Number theory (New York, 1984–1985)*, volume 1240 of *Lecture Notes in Math.*, pages 275–280. Springer, Berlin, 1987.

[26] Jerrold B. Tunnell. A classical Diophantine problem and modular forms of weight 3/2. *Invent. Math.*, 72(2):323–334, 1983.

[27] Jean-Loup Waldspurger. Sur les coefficients de Fourier des formes modulaires de poids demi-entier. *J. Math. Pures Appl. (9)*, 60(4):375–484, 1981.

[28] Andrew Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.

Department of Mathematics, University of South Carolina, Columbia, SC 29208
E-mail address: `jonesmc8@gmail.com`

Department of Mathematics, Wake Forest University, Winston-Salem, NC 27109
E-mail address: `rouseja@wfu.edu`