Fibinomial Identities

Arthur T. Benjamin, Jennifer J. Quinn, and Jeremy A. Rouse

February 24, 2002

Problem $A - 6$ from the 1990 Putnam exam states:

> If X is a finite set, let $|X|$ denote the number of elements in X. Call an ordered pair (S, T) of subsets of $\{1, 2, \ldots, n\}$ *admissible* if $s > |T|$ for each $s \in S$, and $t > |S|$ for each $t \in T$. How many admissible ordered pairs of subsets of $\{1, 2, \ldots, 10\}$ are there? Prove your answer.

It is no coincidence that the solution, 17711, is the 21st Fibonacci number. The number of admissible ordered pairs of subsets of $\{1, 2, \ldots, n\}$ with $|S| = a$ and $|T| = b$ is $\binom{n-b}{a}\binom{n-a}{b}$. Summing over all values of a and b leads to

Identity 1

$$\sum_{a=0}^{n}\sum_{b=0}^{n}\binom{n-b}{a}\binom{n-a}{b} = f_{2n+1}.$$

where $f_0 = 1$, $f_1 = 1$ and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. The published solutions [6, 10] use "convoluted" algebraic methods. Yet the presence of both Fibonacci numbers and binomial coefficients demands a combinatorial explanation. Beginning with our proof of Identity 1, we provide simple, combinatorial arguments for many *fibinomial identities* — identities that combine (generalized) Fibonacci numbers and binomial coefficients.

Fibonacci numbers can be combinatorially interpreted in many ways [9]. The primary tool used in this note will be tilings of $1 \times n$ boards with tiles of varying lengths. The identities presented are viewed as counting questions, answered in two different ways. To begin with, Identity 1 is easily seen by answering

Question: How many ways can a board of length $2n+1$ be tiled using (length 1) squares and (length 2) dominoes?

Answer 1: A length m board can be tiled in f_m ways, which can be seen by conditioning on whether the last tile is a square or a domino. Consequently, a board of length $2n + 1$ can be tiled f_{2n+1} ways.

Answer 2: Condition on the number of dominoes on each side of the *median* square.

Any tiling of a $(2n + 1)$-board must contain an odd number of squares. Thus one square, which we call the median square, contains an equal number of squares to the left and right of it. For example, the 13-tiling in Figure 1 has 5 squares. The median square, the third square, is located in cell 9.

How many tilings contain exactly a dominoes to the left of the median square and exactly b dominoes to the right of the median square? Such a tiling has $(a + b)$ dominoes and therefore $(2n + 1) - 2(a + b)$ squares. Hence the median square has $n - a - b$ squares on each side of it. Since the left side has $(n - a - b) + a = n - b$ tiles, of which a are dominoes, there are $\binom{n-b}{a}$ ways to tile to the left of the median square. Similarly, there are $\binom{n-a}{b}$ ways to tile to the right of the median square. Hence there are $\binom{n-b}{a}\binom{n-a}{b}$ tilings altogether.

Varying a and b overall feasible values, we obtain the total number of $(2n + 1)$-tilings as $\sum_{a=0}^{n}\sum_{b=0}^{n}\binom{n-b}{a}\binom{n-a}{b}$.

median square

Figure 1: Every square-domino tiling of odd length must have a median square. The 13-tiling above has 3 dominoes left of the median square and 1 domino to the right of the median square. The number of such tilings is $\binom{5}{3}\binom{3}{1}$.

We can extend this identity by utilizing the 3-bonacci numbers, defined by $\theta_n = 0$ for $n < 0$, $\theta_0 = 1$ and for $n \geq 1$, $\theta_n = \theta_{n-1} + \theta_{n-3}$.

Identity 2 $\sum_{a=0}^{n} \sum_{b=0}^{n} \sum_{c=0}^{n} \binom{n-b-c}{a} \binom{n-a-c}{b} \binom{n-a-b}{c} = \theta_{3n+2}$.

Question: How many ways can a board of length $3n + 2$ be tiled using squares and trominoes?

Answer 1: It is easy to see that θ_n counts the number of ways to tile a board of length n with squares and (length 3) trominoes. Hence there are θ_{3n+2} such tilings of a board of length $3n + 2$.

Answer 2: The number of squares in any tiling of a $(3n + 2)$-board must be 2 greater than a multiple of 3. Hence there will exist two *goalpost* squares, say located at cells x and y, such that there are an equal number of squares to the left of x, between x and y, and to the right of y. We condition on the number of trominoes in the three regions defined by the goalposts. If the number of trominoes in each region is, from left to right, a, b, c, then there are a total of $a + b + c$ trominoes and $(3n + 2) - 3(a + b + c)$ squares, including the two goldpost squares. Thus each region has $n - (a + b + c)$ squares. The leftmost region has $n - b - c$ tiles, a of which are trominoes, and there are $\binom{n-b-c}{a}$ ways to arrange them. Likewise the tiles of the second and third region can be arranged $\binom{n-a-c}{b}$ ways and $\binom{n-a-b}{c}$ ways, respectively.

As a, b, and c vary, the total number of $(3n + 2)$-tilings is the left side of our identity.

Applying the same logic, we immediately obtain the following k-bonacci generalization.

Identity 3 *Let κ_n be the k-bonacci number defined for $n \geq 1$ by $\kappa_n = \kappa_{n-1} + \kappa_{n-k}$, where $\kappa_0 = 1$ and for $j < 0$, $\kappa_j = 0$. Then for $n \geq 0$, $\kappa_{kn+(k-1)}$ equals*

$$\sum_{a_1=0}^{n} \sum_{a_2=0}^{n} \cdots \sum_{a_k=0}^{n} \binom{n - (a_2 + a_3 + \cdots + a_k)}{a_1} \binom{n - (a_1 + a_3 + \cdots + a_k)}{a_2} \cdots \binom{n - (a_1 + a_2 + \cdots + a_{k-1})}{a_k}.$$

Another generalization of Fibonacci numbers are the k-th order Fibonacci numbers defined by $g_n = 0$ for $n < 0$, $g_0 = 1$, and for $n \geq 1$, $g_n = g_{n-1} + g_{n-2} + \cdots + g_{n-k}$. The next identity is proved by non-trivial algebraic methods in [7] and [8], but when viewed combinatorially, as done in [5], it is practically obvious.

Identity 4 *For all $n \geq 0$,*

$$g_n = \sum_{n_1} \sum_{n_2} \cdots \sum_{n_k} \binom{n_1 + n_2 + \cdots + n_k}{n_1, n_2, \ldots, n_k},$$

where the summation is over all non-negative integers $n_1, n_2, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n$.

Question: In how many ways can we tile a board of length n using tiles with lengths at most k?

Answer 1: By its definition, it is combinatorially clear that g_n counts this quantity.

Answer 2: Condition on the number of tiles of each length. If for $1 \leq i \leq k$ there are n_i tiles of length i, then we must have $n_1 + 2n_2 + \cdots + kn_k = n$. The number of ways to permute these tiles is given by the multinomial coefficient $\binom{n_1+n_2+\cdots+n_k}{n_1,n_2,\ldots,n_k}$.

2

More colorfully, for nonnegative integers $c_1, \ldots, c_k$ we define the generalized k-th order Fibonacci number by $h_n = 0$ for $n < 0$, $h_0 = 1$, and for $n \geq 1$, $h_n = c_1 \cdot h_{n-1} + c_2 \cdot h_{n-2} + \cdots + c_k \cdot h_{n-k}$. It is easy to see that h_n counts the number of ways to tile a board of length n with *colored* tiles of length at most k, where for $1 \leq i \leq k$, a tile of length i may be assigned any one of c_i colors. The previous identity and argument immediately generalizes to the following identity.

Identity 5 *For all $n \geq 0$,*

$$h_n = \sum_{n_1} \sum_{n_2} \cdots \sum_{n_k} \binom{n_1 + n_2 + \cdots + n_k}{n_1, n_2, \ldots, n_k} c_1^{n_1} c_2^{n_2} \cdots c_k^{n_k},$$

where the summation is over all non-negative integers $n_1, n_2, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n$.

Identity 1 can be generalized another way. The following identity originally appeared in [4] and was proved by sophisticated algebraic methods.

Identity 6

$$\sum_{a_1=0}^{n} \sum_{a_2=0}^{n} \cdots \sum_{a_k=0}^{n} \binom{n - a_k}{a_1} \binom{n - a_1}{a_2} \cdots \binom{n - a_{k-1}}{a_k} = f_{kn+k-1}/f_{k-1}.$$

Not surprisingly, this can also be proved combinatorially, [3], although the argument is tricky. What follows are identities with similar appearance that are practically obvious when viewed combinatorially.

Identity 7

$$\sum_{a_1=0}^{n} \sum_{a_2=0}^{n} \cdots \sum_{a_k=0}^{n} \binom{n}{a_1} \binom{n - a_1}{a_2} \cdots \binom{n - a_{k-1}}{a_k} = f_{k+1}^n.$$

Question: In how many ways can we simultaneously tile n distinguishable boards of length $k + 1$ with squares and dominoes?

Answer 1: Since each board can be tiled f_{k+1} ways, there are f_{k+1}^n such tilings.

Answer 2: Condition on the number of dominoes covering each consecutive pair of cells. Specifically, we claim there are $\binom{n}{a_1}\binom{n-a_1}{a_2}\cdots\binom{n-a_{k-1}}{a_k}$ ways to create n tilings of length $k + 1$ where a_1 of them begin with dominoes, a_2 have dominoes covering cells 2 and 3, and generally for $1 \leq i \leq k$, a_i of them have dominoes covering cells i and $i + 1$. To see this, notice there are $\binom{n}{a_1}$ ways to decide which of the n tilings begin with a domino (the rest begin with a square). Once these are selected, then among those $n - a_1$ tilings that do not begin with a domino there are $\binom{n-a_1}{a_2}$ ways to determine which of those will have a domino in cells 2 and 3. (The other $n - a_1 - a_2$ tilings will have a square in cell 2.) Continuing in this fashion, we see that once the tilings with dominoes covering cells $i - 1$ and i are determined, there are $\binom{n-a_{i-1}}{a_i}$ ways to determine which tilings have dominoes covering cells i and $i + 1$.

More generally, by tiling n distinguishable boards of length $k + 1$ with squares and dominoes where the first c of them must begin with a square, the same reasoning establishes:

Identity 8 *For $0 \leq c \leq n$,*

$$\sum_{a_1=0}^{n} \sum_{a_2=0}^{n} \cdots \sum_{a_k=0}^{n} \binom{n - c}{a_1} \binom{n - a_1}{a_2} \cdots \binom{n - a_{k-1}}{a_k} = f_k^c f_{k+1}^{n-c}.$$

3

It might surprise you that Identity 7 can be modified to produce

Identity 9

$$\sum_{a_1=0}^{n} \sum_{a_2=0}^{n} \cdots \sum_{a_k=0}^{n} \binom{n}{a_1}\binom{n-a_1}{a_2}\cdots\binom{n-a_{k-1}}{a_k}2^{a_1} = L_{k+1}^n,$$

where L_j is the j-th Lucas number, and more generally

Identity 10

$$\sum_{a_1=0}^{n} \sum_{a_2=0}^{n} \cdots \sum_{a_k=0}^{n} \binom{n}{a_1}\binom{n-a_1}{a_2}\cdots\binom{n-a_{k-1}}{a_k}G_0^{a_1}G_1^{n-a_1} = G_{k+1}^n,$$

where G_j is the j-th *Gibonacci number*, defined by G_0, G_1, and the Fibonacci recurrence $G_j = G_{j-1} + G_{j-2}$ for $j \geq 2$. For non-negative integers G_0 and G_1, G_j can be combinatorially defined [2] as the number of ways to tile a length n board with squares and dominoes subject to the *initial conditions* that the first tile is given a *phase*, where there are G_0 choices for the phase of a domino and G_1 choices for the phase of a square. Applying the logic of the proofs of Identities 7 and 8 gives us the above identity as well as

Identity 11 *For $0 \leq c \leq n$,*

$$\sum_{a_1=0}^{n} \sum_{a_2=0}^{n} \cdots \sum_{a_k=0}^{n} \binom{n-c}{a_1}\binom{n-a_1}{a_2}\cdots\binom{n-a_{k-1}}{a_k}G_0^{a_1}G_1^{n-a_1} = G_1^c f_k^c G_{k+1}^{n-c}.$$

As with Identity 5, these identities can also be generalized by colorizing them.

We end with an open question. The *Fibonomial Numbers* are defined like binomial coefficients with $F's$ on top. That is, Fibonimial $\binom{n}{k}_{\mathcal{F}} = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k}$, where $F_j = f_{j-1}$ is the traditional j-th Fibonacci number. Amazingly $\binom{n}{k}_{\mathcal{F}}$ is always an integer [1]. We challenge the reader to find a combinatorial proof of this fact.

References

[1] G.L. Alexanderson and L.F. Klosinski, A Fibonacci analogue of Gaussian binomial coefficients, *Fibonacci Quarterly* **12** (1974) 129-132.

[2] A.T. Benjamin, J.J. Quinn, and F.E. Su, Generalized Fibonacci Identities through Phased Tilings, *The Fibonacci Quarterly*, Vol. 38 No. 3 (2000) 282-288.

[3] A.T. Benjamin and J.A. Rouse, Recounting binomial Fibonacci identities, *The Fibonacci Quarterly* to appear.

[4] L. Carlitz, The characteristic polynomial of a certain matrix of binomial coefficients, *The Fibonacci Quarterly* **3.2** (1965) 81-89.

[5] C.R.H. Hanusa, A.T. Benjamin, and F.E. Su, Linear recurrences through tilings and Markov chains, *Utilitas Mathematica* to appear.

[6] L.F. Klosinski, G.L. Alexanderson, L.C. Larson, The Fifty-First William Lowell Putnam Mathematical Competition, *American Mathematical Monthly* **98**(8), (1991) 719-727.

[7] A.N. Philippou, A note on the Fibonacci sequence of order K and the multinomial coefficients, *Fibonacci Quart.* **21**(2), (1983)82-86.

[8] A.N. Philippou and A.A. Muwafi, Waiting for the K-th consecutive success and the Fibonacci sequence of order K, *Fibonacci Quart.* **20**(1), (1982) 28-32.

[9] R.P. Stanley, *Enumerative Combinatorics, Vol. 1*, Wadsworth & Brooks/Cole. Chapter 1 problem 14 a-i.

[10] The citation for solutions in Mathematics Magazine? I can't find it. How about you?