ON SPACES OF MODULAR FORMS SPANNED BY ETA-QUOTIENTS

JEREMY ROUSE AND JOHN J. WEBB

Abstract. An eta-quotient of level N is a modular form of the shape $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$. We study the problem of determining levels N for which the graded ring of holomorphic modular forms for $\Gamma_0(N)$ is generated by (holomorphic, respectively weakly holomorphic) eta-quotients of level N. In addition, we prove that if $f(z)$ is a holomorphic modular form that is non-vanishing on the upper half plane and has integer Fourier coefficients at infinity, then $f(z)$ is an integer multiple of an eta-quotient. Finally, we use our results to determine the structure of the cuspidal subgroup of $J_0(2^k)(\mathbb{Q})$.

1. Introduction

In 1877, Richard Dedekind [3] defined the function

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i z}, \tag{1}$$

for $z \in \mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$, which is now known as the Dedekind eta-function. The reciprocal of the Dedekind eta-function is

$$\frac{1}{\eta(z)} = q^{-1/24} \sum_{n=0}^{\infty} p(n) q^n,$$

where $p(n)$ is the number of partitions of n. The modular transformations of the Dedekind eta-function, which take the form

$$\eta\left(\frac{az+b}{cz+d}\right) = \epsilon(a,b,c,d)(cz+d)^{1/2}\eta(z) \text{ for } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } \epsilon(a,b,c,d)^{24} = 1,$$

play an important role in Hardy and Ramanujan's exact formula for $p(n)$, derived using the circle method.

An *eta-quotient* of level N is a function of the shape

$$f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta},$$

2010 *Mathematics Subject Classification.* Primary 11F20, 11F30; Secondary 11G18.

where $r_\delta \in \mathbb{Z}$. As a consequence of the product definition for $\eta(z)$, any eta-quotient is non-vanishing on $\mathbb{H}$. The following are examples of well-known eta-quotients:

$$\Delta(z) = \eta(z)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n , \quad \frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2} = \sum_{n \in \mathbb{Z}} q^{n^2} , \quad \frac{\eta(4z)^8}{\eta(2z)^4} = \sum_{n=0}^{\infty} \sigma_1(2n+1) q^{2n+1},$$

where for all positive integers j, t, we have $\sigma_t(j) = \sum_{d|j} d^t$.

Let

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \text{ and } N|c \right\}.$$

A weight k weakly holomorphic modular form is a function on $\mathbb{H}$ that obeys the weight k modular transformation law for $\Gamma_0(N)$, is holomorphic on $\mathbb{H}$, but may possess poles at the cusps. In the 1950s, Morris Newman (see [12] and [13]) used the Dedekind eta-function to systematically build weakly holomorphic modular forms for $\Gamma_0(N)$. A key ingredient in his work is a classification of when $f(z)$ is actually a modular form for $\Gamma_0(N)$. If

$$(2) \qquad \sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}, \quad \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}, \text{ and}$$
$$\prod_{\delta|N} \delta^{r_\delta} \text{ is the square of a rational number,}$$

then $f(z)$ transforms like a weight $k = \frac{1}{2}\sum_{\delta|N} r_\delta$ modular form for $\Gamma_0(N)$. We denote the set of weight k modular forms on $\Gamma_0(N)$ by $M_k(\Gamma_0(N))$.

Let $Y_0(N)$ be the algebraic curve $\mathbb{H}/\Gamma_0(N)$, and let $X_0(N)$ be its compactification. This algebraic curve has a model defined over $\mathbb{Q}$ given by the classical modular equation $\Phi_N(x, y) = 0$ (see for example [16], pg. 109-110). In [8], Ligozat determines the order of vanishing of an eta quotient at the cusps of the level N modular curve $X_0(N)$.

Theorem. *Let c, d and N be positive integers with $d|N$ and $\gcd(c, d) = 1$. If $f(z)$ is an eta-quotient, then the order of vanishing of $f(z)$ at the cusp c/d is*

$$\frac{N}{24} \sum_{\delta|N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, N/d) d\delta}.$$

Remark. *The formula for the order of vanishing only depends on d, and not on c.*

In [14], Ken Ono observes that every holomorphic modular form for $\mathrm{SL}_2(\mathbb{Z})$ can be expressed as a linear combination of eta-quotients of level 4, and poses the following problem: "Classify the spaces of modular forms which are generated by eta-quotients." The goals of the present paper are to address this question, to give an intrinsic characterization of level N eta-quotients, and to apply this information to the study of the cuspidal subgroup of the Jacobian of $X_0(N)$.

2. Generating spaces with holomorphic and weakly holomorphic eta-quotients

Combining the hypotheses in the theorems of Newman and Ligozat show that the sequences of integers $\{r_\delta : \delta | N\}$, which are the exponents of an eta-quotient in $M_k(\Gamma_0(N))$, correspond to integer points inside a $d(N)-1$-dimensional convex polytope, where $d(N)$ is the number of divisors of N.

Example. Using the lattice point enumeration program LattE [1], we are able to determine precisely the number of eta-quotients in $M_k(\Gamma_0(36))$ for various values of k. These eta-quotients correspond to lattice points in an 8-dimensional polytope whose volume increases quickly with k. There are 4,988 eta-quotients in $M_2(\Gamma_0(36))$ and there are 703,060,312 eta-quotients in $M_{12}(\Gamma_0(36))$.

Theorem 1. *There are precisely* 121 *positive integers* $N \leq 500$ *so that the graded ring of modular forms for* $\Gamma_0(N)$ *is generated by eta-quotients.*

A necessary condition for eta-quotients to generate the graded ring of modular forms on $\Gamma_0(N)$ if N is composite is for $M_2(\Gamma_0(N))$ to be spanned by eta-quotients. Numerical evidence suggests that it is only possible to find enough eta-quotients to span $M_2(\Gamma_0(N))$ if N is sufficiently composite. We make this precise in our next result.

Theorem 2. *Suppose that* $f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta} \in M_k(\Gamma_0(N))$. *Then we have*

$$\sum_{\delta | N} |r_\delta| \leq 2k \prod_{p | N} \left(\frac{p+1}{p-1}\right)^{\min\{2, \mathrm{ord}_p(N)\}}.$$

Remark. *The bound is sharp. For each* N, *there is a weight* k *and an eta-quotient* $f(z) \in M_k(\Gamma_0(N))$ *for which the bound is achieved.*

If k and the number of divisors of N are fixed, the number of sequences of integers r_δ that satisfy the inequality in Theorem 2 is bounded. Since $\dim M_k(\Gamma_0(N))$ tends to infinity with N, it follows that for a fixed k and a given number of prime divisors of N, there are only finitely many spaces $M_k(\Gamma_0(N))$ that are spanned by eta-quotients. Accordingly, a space $M_k(\Gamma_0(N))$ will be spanned by eta-quotients only if N is "sufficiently composite" in relation to its size.

Remark. *Theorem 2 and the consequences regarding* $M_k(\Gamma_0(N))$ *being spanned by eta-quotients were first obtained by Soumya Bhattacharya in 2011 who is (as of this writing) a Ph.D. student of Professor Don Zagier at the University of Bonn. Bhattacharya's work is not yet available in manuscript form.*

Our next result gives examples of spaces that are not spanned by eta-quotients.

Corollary 3. *If* p *is prime, then* $M_2(\Gamma_0(4p))$ *is spanned by eta-quotients if and only if* $p \leq 13$.

Instead of insisting that every element of $M_k(\Gamma_0(N))$ be expressible as a linear combination of eta-quotients which are holomorphic on $\mathbb{H}$, we could instead ask if every element of $M_k(\Gamma_0(N))$ is expressible as a linear combination of *weakly holomorphic* eta-quotients of weight k and level N with a pole only at infinity.

For example, $M_2(\Gamma_0(22))$ has dimension 5, but only contains 4 eta-quotients, namely

$$\frac{\eta(2z)^4\eta(22z)^4}{\eta(z)^2\eta(11z)^2}, \eta(z)^2\eta(11z)^2, \eta(2z)^2\eta(22z)^2, \text{ and } \frac{\eta(z)^4\eta(11z)^4}{\eta(2z)^2\eta(22z)^2}.$$

The basis element $f(z) = q^4+q^6+q^8+q^{10}-q^{11}+\cdots$ is not expressible as a linear combination of these four. However, if $g(z) = \frac{\eta(22z)^{22}\eta(z)}{\eta(2z)^2\eta(11z)^{11}}$, then $f(z)g(z) \in M_{12}(\Gamma_0(22))$ and every holomorphic modular form in $M_{12}(\Gamma_0(22))$ is a linear combination of (holomorphic) eta-quotients. Since $g(z)$ is non-vanishing at all cusps except infinity, this implies that $f(z)$ is a linear combination of weakly holomorphic eta-quotients with poles only at infinity.

Next, we will study the levels N for which every form in $M_k(\Gamma_0(N))$ is a linear combination of weakly holomorphic eta-quotients with a pole only at infinity. To state our result, let $\mathcal{R}_k(\Gamma_0(N))$ be the vector space of all weakly holomorphic modular forms that have a pole only at the cusp at infinity. Let $\mathcal{E}_k(\Gamma_0(N))$ be the subspace of $\mathcal{R}_k(\Gamma_0(N))$ consisting of forms that are linear combinations of eta-quotients with a pole only at infinity. Note that $\mathcal{R}_0(\Gamma_0(N))$ and $\mathcal{E}_0(\Gamma_0(N))$ are rings, and $\mathcal{R}_k(\Gamma_0(N))$ and $\mathcal{E}_k(\Gamma_0(N))$ naturally have the structure of modules over $\mathcal{R}_0(\Gamma_0(N))$ and $\mathcal{E}_0(\Gamma_0(N))$, respectively.

Our next result is the following. Let $\epsilon_2(\Gamma_0(N))$ and $\epsilon_3(\Gamma_0(N))$ denote the number of orbits of elliptic points of order 2 and 3 for $\Gamma_0(N)$, respectively.

Theorem 4. (1) *Suppose that N is composite or $N \in \{2, 3, 5, 7, 13\}$. Suppose also that either $\epsilon_3(\Gamma_0(N)) = 0$ or $6|k$, and either $\epsilon_2(\Gamma_0(N)) = 0$ or $4|k$. Then $\mathcal{E}_k(\Gamma_0(N))$ has finite codimension in $\mathcal{R}_k(\Gamma_0(N))$.*
(2) *If N is prime and $N = 11$ or $N \geq 17$, then $\mathcal{E}_k(\Gamma_0(N))$ has infinite codimension in $\mathcal{R}_k(\Gamma_0(N))$ for all non-negative even integers k.*

Our next result gives a classification of the levels N for which every element of $M_k(\Gamma_0(N))$ has an expression in terms of weakly holomorphic eta-quotients.

Theorem 5. *Let N be a positive integer. Then the following are equivalent.*

(1) *For all positive even integers k, $M_k(\Gamma_0(N)) \subset \mathcal{R}_k(\Gamma_0(N))$. In other words, every element of $M_k(\Gamma_0(N))$ is expressible as a linear combination of weakly holomorphic weight k eta-quotients of level N with poles only at infinity.*
(2) *$\mathcal{R}_k(\Gamma_0(N)) = \mathcal{E}_k(\Gamma_0(N))$ for all non-negative even integers k.*
(3) *$\mathcal{R}_2(\Gamma_0(N)) = \mathcal{E}_2(\Gamma_0(N))$.*

Theorem 4 and the proof of Theorem 5 show that in order for $\mathcal{R}_2(\Gamma_0(N))$ to equal $\mathcal{E}_2(\Gamma_0(N))$, it is necessary for N to be composite and for $\Gamma_0(N)$ to have no elliptic points. Our next result shows that for most small N, this is sufficient.

Theorem 6. *If $N \leq 300$ is composite, $\Gamma_0(N)$ has no elliptic points, and $N \notin \{121, 209\}$, then $\mathcal{R}_2(\Gamma_0(N)) = \mathcal{E}_2(\Gamma_0(N))$.*

Remark. *It appears likely that for $N = 121$ and $N = 209$, the equivalent conditions of Theorem 5 are false.*

There are two significant consequences of being able to span spaces of modular forms using eta-quotients. First, this provides a means of computing the Fourier expansions of a basis for $M_k(\Gamma_0(N))$. The theory of modular symbols can also be used to do this, but this process is somewhat inefficient in that it is first necessary to compute the Fourier expansions of Hecke eigenforms, and then to build a basis for $M_k(\Gamma_0(N))$ using those. There are a number of situations where we are able to quickly construct bases for spaces $M_k(\Gamma_0(N))$ with dimension greater than 2000 where the modular symbols algorithm is much less efficient. Second, if $f(z) \in M_k(\Gamma_0(N))$, it is sometimes desirable to compute the Fourier expansion of $f(z)$ at other cusps of $\Gamma_0(N)$. There are no general algorithms to accomplish this task. However, if $f(z)$ is an eta-quotient (or a linear combination of eta-quotients), it is straightforward to do this, since the transformation formula for $\eta(z)$ is known for every matrix in $\mathrm{SL}_2(\mathbb{Z})$.

3. Modular forms non-zero on $\mathbb{H}$

Looking at (1), it is not difficult to see that any eta-quotient must have integral Fourier coefficients and be non-zero on the upper-half plane. We will examine to what degree the converse is true. Kohnen [7] studied a related problem, which we now describe. Let

$$f(z) = cq^h \prod_{n \geq 1} (1 - q^n)^{c(n)} \in M_k^!(\Gamma_0(N))$$

where $M_k^!(\Gamma_0(N))$ denotes the space of weight k weakly holomorphic modular forms on $\Gamma_0(N)$, c is a non-zero constant, $h \in \mathbb{Z}$ and each $c(n) \in \mathbb{C}$. Then he proves the following.

Theorem (Theorem 2 in [7])**.** *Suppose that N is square-free. Then f has no zeros or poles on $\mathbb{H}$ if and only if $c(n)$ depends only on the greatest common divisor (n, N).*

A direct consequence of his proof is that for some $f(z) \in M_k^!(\Gamma_0(N))$ that has no zeros or poles on $\mathbb{H}$ and where N-square free, there exists some positive integer t such that $f(z)^t$ is a constant times an eta-quotient. If we replace the condition that N is square-free with $f(z) \in \mathbb{Z}[[q]]$, we prove the following result.

Theorem 7. *Suppose $f(z) \in M_k(\Gamma_0(N)) \cap \mathbb{Z}[[q]]$ has the property that f is non-zero on $\mathbb{H}$. Then $f(z) = cg(z)$ where $c \in \mathbb{Z}$ and $g(z)$ is an eta-quotient.*

Let κ be an even integer and $g(z) \in M^{!}_{\kappa}(\Gamma_0(N)) \cap \mathbb{Z}((q))$ be non-zero on $\mathbb{H}$. Since $\Delta(z) = \eta(z)^{24}$ is zero at every cusp of $\Gamma_0(N)$, for sufficiently large t, $g(z)\Delta(z)^t$ is a holomorphic modular form on $\Gamma_0(N)$ with integral Fourier coefficients and non-zero on the upper-half plane. By Theorem 7, this implies $g(z)\Delta(z)^t$ is a constant times an eta-quotient which then implies the following.

Corollary 8. *Suppose $g(z) \in M^{!}_{\kappa}(\Gamma_0(N)) \cap \mathbb{Z}((q))$ has the property that g is non-zero on $\mathbb{H}$. Then $g(z) = ch(z)$ where $c \in \mathbb{Z}$ and $h(z)$ is an eta-quotient.*

On the other hand, the condition $f(z) \in \mathbb{Z}[[q]]$ in Theorem 7 is necessary. To see this, suppose that p is an odd prime with $p^2 \mid N$. Then by the a theorem of Manin and Drinfeld [5], [10], the cuspidal subgroup of the Jacobian of $X_0(N)$ is finite which implies that there exists a $g(z) \in M^{!}_{0}(\Gamma_0(N))$ whose zeros are only supported at the cusp $1/p$ and whose poles are only supported at the cusp ∞. It follows that for a sufficiently large t, we have $f(z) = g(z)\Delta(z)^t \in M_{12t}(\Gamma_0(N))$. Such a form is non-zero on $\mathbb{H}$. Since $\Delta(z) = \eta(z)^{24}$, the remark following the theorem of Ligozat implies that $\mathrm{ord}_{\frac{1}{p}}\Delta(z) = \mathrm{ord}_{\frac{2}{p}}\Delta(z)$. We also have $\mathrm{ord}_{\frac{1}{p}} g(z) > 0$ and $\mathrm{ord}_{\frac{2}{p}} g(z) = 0$. It follows that $\mathrm{ord}_{\frac{1}{p}}(f(z)) > \mathrm{ord}_{\frac{2}{p}}(f(z))$. Using the remark after the theorem of Ligozat again, we conclude that $f(z)$ is not an eta-quotient. By Corollary 8, such a modular function $g(z)$ must have non-integral Fourier coefficients. Another way of seeing this it to note that the cusps $\frac{1}{p}$ and $\frac{2}{p}$ on $X_0(N)$ are not defined over $\mathbb{Q}$ and are in the same orbit under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Therefore, a modular form with a zero at one and no zero at the other cannot be fixed by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and this implies it cannot have rational Fourier coefficients.

The multiplicative group of $M^{!}_{0}(\Gamma_0(N))$ is often called the modular units on $\Gamma_0(N)$. If $f(z)$ is a modular unit, then by definition $f(z)^{-1} \in M^{!}_{0}(\Gamma_0(N))$ as well, which implies that f must be non-zero on $\mathbb{H}$. Consider $\mathcal{M}(N) = M^{!}_{0}(\Gamma_0(N))^{\times} \cap \mathbb{Z}((q))^{\times}$. This is the subgroup of the modular units on $\Gamma_0(N)$ with integral coefficients and a leading coefficient of ± 1 . A special case of Corollary 8 is the following.

Corollary 9. *Suppose $g(z) \in \mathcal{M}(N)$. Then $g(z) = \pm f(z)$ where $f(z)$ is an eta-quotient.*

Our next result concerns the structure of the rational cuspidal subgroup $J_0(N)(\mathbb{Q})_{\mathrm{cusp}}$ when N is a power of 2. Let $\mathrm{Div}^0(X_0(N)/\mathbb{Q})$ denote the degree 0 divisors of the modular curve $X_0(N)$ fixed by every element of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then $J_0(N)(\mathbb{Q})_{\mathrm{cusp}}$ is isomorphic to $\mathrm{Div}^0(X_0(N)/\mathbb{Q})/\,\mathrm{div}(\mathcal{M}(N))$. Ling [9] computed the structure of $J_0(p^k)(\mathbb{Q})_{\mathrm{cusp}}$ where $p \geq 3$ is prime and $k \geq 1$. Using Corollary 9, we compute $J_0(2^k)(\mathbb{Q})_{\mathrm{cusp}}$. Let $k \geq 1$ and define the set I_k as follows. For $1 \leq k \leq 6$, define

$$I_k = \begin{cases} \emptyset & \text{if } 1 \leq k \leq 4\,, \\ \{2\} & \text{if } k = 5\,, \\ \{1, 2, 2\} & \text{if } k = 6\,. \end{cases}$$

For $k \geq 7$, if k is odd, define

$$i_{t,k} = \begin{cases} \lfloor \frac{t-1}{2} \rfloor + \frac{k-1}{2} - 2 & \text{if } 1 \leq t \leq k-1\,, \\ k-3 & \text{if } t = k-2\,, \end{cases}$$

and if k is even, define

$$i_{t,k} = \begin{cases} \lfloor \frac{t}{2} \rfloor + \frac{k}{2} - 3 & \text{if } 1 \leq t \leq k-4\,, \\ k-4 & \text{if } t = k-3 \text{ or } k-2\,. \end{cases}$$

Then let

$$I_k = \{i_{t,k}\}_{t=1}^{k-2}\,.$$

Theorem 10. *For $k \geq 1$,*

$$J_0(2^k)(\mathbb{Q})_{cusp} \cong \bigoplus_{i \in I_k} \mathbb{Z}/2^i\mathbb{Z}\,.$$

We now give a formula for the number of weight k eta-quotients of level N assuming that $X_0(N)$ has genus zero (and hence $J_0(N)(\mathbb{Q})_{\mathrm{cusp}}$ is trivial).

Theorem 11. *Assume that N is a positive integer so that $X_0(N)$ has genus zero, and there is a holomorphic eta-quotient in $M_k(\Gamma_0(N))$. Then, the number of eta-quotients in $M_k(\Gamma_0(N))$ is equal to the number of tuples of non-negative integers $(c_d : d|N)$ with*

$$\sum_{d|N} c_d \phi\left(\gcd\left(d, \frac{N}{d}\right)\right) = \frac{k}{12} \cdot [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)].$$

Here $\phi(n)$ is the usual Euler totient function.

Remark. *As a consequence, there are $\frac{k^2}{8} + \frac{3k}{4} + 1$ eta-quotients in $M_k(\Gamma_0(4))$, $\frac{k^3}{6} + k^2 + \frac{11k}{6} + 1$ eta-quotients in $M_k(\Gamma_0(8))$, and $\frac{k^4}{3} + 2k^3 + \frac{25k^{2]}}{6} + \frac{7k}{2} + 1$ eta-quotients in $M_k(\Gamma_0(16))$.*

Acknowledgments We used Magma [2] version 2.19-8 and LattE [1] version 1.5 for computations. Details about the computations that were done are available at `http://users.wfu.edu/rouseja/eta/`. The authors wish to thank the anonymous referees for helpful comments which have improved the exposition.

4. Background

Given a prime number p and a non-zero integer n, we define $\mathrm{ord}_p(n)$ to be the highest power of p that evenly divides n.

For a positive even integer k, let

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

be the usual weight k Eisenstein series, where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the sum of the $k-1$st powers of the divisors of n. Here B_k is the kth Bernoulli number. If $k \geq 4$, then $E_k(z) \in M_k(\Gamma_0(1))$. Let

$$j(z) = \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + 744q + 196884q + \cdots$$

be the usual modular j-function. If d is a positive integer, define the operator $V(d)$ by

$$f(z) = \left(\sum_{n=0}^{\infty} a(n)q^n\right) |V(d) = f(dz) = \sum_{n=0}^{\infty} a(n)q^{dn}.$$

It is well-known that if $f(z) \in M_k(\Gamma_0(N))$, then $f(z)|V(d) \in M_k(\Gamma_0(dN))$. (See for example Proposition 2.22 of [14].)

A level N modular function is a meromorphic modular form of weight zero for $\Gamma_0(N)$. It is known that a level N modular function is a rational function in $j(z)$ and $j(Nz)$. (See Proposition 7.5.1 of Diamond and Shurman's book [4], page 279.)

An elliptic point of $\Gamma_0(N)$ is a number $z \in \mathbb{H}$ so that there is an $M \in \Gamma_0(N)$ with $Mz = z$ and $M \neq \pm I$. In this case M has order 2 or 3 in $\mathrm{PSL}_2(\mathbb{Z})$, and we call z an elliptic point of order 2 or 3. Let $\epsilon_2(\Gamma_0(N))$ denote the number of $\Gamma_0(N)$ orbits of elliptic points of order 2, and $\epsilon_3(\Gamma_0(N))$ the number of orbits of elliptic points of order 3. Diamond and Shurman ([4], Corollary 3.7.2, page 96) give the formulas

$$\epsilon_2(\Gamma_0(N)) = \begin{cases} \prod_{p|N} \left(1 + \left(\frac{-1}{p}\right)\right) & \text{if } 4 \nmid N \\ 0 & \text{if } 4|N \end{cases}$$

$$\epsilon_3(\Gamma_0(N)) = \begin{cases} \prod_{p|N} \left(1 + \left(\frac{-3}{p}\right)\right) & \text{if } p \nmid N \\ 0 & \text{if } 9|N. \end{cases}$$

The action of $\Gamma_0(N)$ on the upper half-plane extends to an action on $\mathbb{P}^1(\mathbb{Q})$. The cusps of $\Gamma_0(N)$ are the $\Gamma_0(N)$-orbits of $\mathbb{P}^1(\mathbb{Q})$. Given a number $(a : c) \in \mathbb{P}^1(\mathbb{Q})$ (with $a, c \in \mathbb{Z}$ and $\gcd(a, c) = 1$), there is a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $M(\infty) = a/c$. If $f \in M_k(\Gamma_0(N))$, the order of vanishing of f at the cusp a/c is the smallest power of q with a nonzero coefficient in the expansion of

$$(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) = \sum_{n=0}^{\infty} a(n)q^{n/N}.$$

This will be denoted by $\mathrm{ord}_{a/c}(f)$. It does not depend on the matrix M chosen, and if $(a : c)$ and $(a' : c')$ are $\Gamma_0(N)$-equivalent, then $\mathrm{ord}_{a/c}(f) = \mathrm{ord}_{a'/c'}(f)$.

The graded ring of modular forms for $\Gamma_0(N)$ is $\bigoplus_{\substack{k \geq 0 \\ k \text{ even}}} M_k(\Gamma_0(N))$. We require some results about the weights in which this graded ring is generated. Let $\mathcal{L}$ be the sheaf of 1-forms on $X_0(N)$. Since $X_0(N)$ is non-singular, $\mathcal{L}$ is invertible and $M_k(\Gamma_0(N))$ is isomorphic to $H^0(X_0(N), \mathcal{L}^{\otimes k/2})$. Multiplication of modular forms corresponds to maps

$$H^0(X_0(N), \mathcal{L}^{\otimes a}) \otimes H^0(X_0(N), \mathcal{L}^{\otimes b}) \to H^0(X_0(N), \mathcal{L}^{\otimes a+b}).$$

In [11], page 55, Mumford shows that if $\mathcal{L}$ is an invertible sheaf on a curve of genus g and with degree $\geq 2g+1$, then L is ample with normal generation. This implies that the tensor product map above is surjective, for all $a, b \geq 1$ and hence the graded ring of modular forms is generated in weight 2. This result seems to have been rediscovered by Rustom [15] and Khuri-Makdisi (see Proposition 2.1 of [6]).

If $\Gamma_0(N)$ has no elliptic points, the degree of the invertible sheaf $\mathcal{L}$ is $2g - 2 + \epsilon_\infty$ where ϵ_∞ is the number of cusps of $\Gamma_0(N)$. If N is composite, then $\epsilon_\infty \geq 3$. Hence, if N is composite we have that the multiplication map $M_k(\Gamma_0(N)) \times M_l(\Gamma_0(N)) \to M_{k+l}(\Gamma_0(N))$ is surjective for all k and l with $k, l \geq 2$. This shows that the graded ring of modular forms is generated in weight 2 as long as N is composite and $\Gamma_0(N)$ has no elliptic points.

5. Proofs

First, we will analyze when there are eta-quotients of weight k for $\Gamma_0(N)$. This is determined by the presence of elliptic points for $\Gamma_0(N)$. Recall that $\mathcal{E}_k(\Gamma_0(N))$ is the space of weakly-holomorphic eta quotients of weight k and level N.

Lemma 12. *Fix a positive integer N. Then for even positive integers k, $\mathcal{E}_k(\Gamma_0(N))$ is non-empty precisely when*

(1) $k \equiv 0 \pmod{12}$ *if* $\epsilon_2(\Gamma_0(N)) > 0$ *and* $\epsilon_3(\Gamma_0(N)) > 0$,
(2) $k \equiv 0 \pmod{6}$ *if* $\epsilon_2(\Gamma_0(N)) = 0$ *and* $\epsilon_3(\Gamma_0(N)) > 0$,
(3) $k \equiv 0 \pmod{4}$ *if* $\epsilon_2(\Gamma_0(N)) > 0$ *and* $\epsilon_3(\Gamma_0(N)) = 0$, *and*
(4) $k \equiv 0 \pmod{2}$ *if* $\epsilon_2(\Gamma_0(N)) = \epsilon_3(\Gamma_0(N)) = 0$.

Moreover, if $\mathcal{E}_k(\Gamma_0(N))$ is non-empty, then there is a holomorphic eta-quotient in $M_k(\Gamma_0(N))$.

Proof. First, suppose that f is a weakly holomorphic modular form of weight k for $\Gamma_0(N)$. If z is an elliptic point for $\Gamma_0(N)$, then $\frac{az+b}{cz+d} = z$ for $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. It follows from this that $|cz + d| = 1$. The transformation law gives that

$$f(z) = (cz + d)^k f(z).$$

Thus, either $(cz+d)^k = 1$ or $f(z) = 0$. Corollary 2.3.4 of [4] (page 55) implies that any elliptic point of order 2 has the form $\frac{ai+b}{ci+d}$ or $\frac{a\omega+b}{c\omega+d}$ and where $\omega = e^{2\pi i/3}$. It follows from this fact that if z is an elliptic point of order 3, then $cz + d = \omega$ or $1 - \omega$, and if z is an elliptic point of order 2 then $cz + d = i$. It follows that if $k \not\equiv 0 \pmod{6}$ and z is an elliptic point of order 3, then $f(z) = 0$. Also, if $k \not\equiv 0 \pmod{4}$ and z is an elliptic point of order 2, then $f(z) = 0$.

Assuming that $f(z)$ is a weakly holomorphic eta-quotient, then we must have that $f(z)$ is non-vanishing on $\mathbb{H}$ and this shows that in order for an eta-quotient to exist, the conditions in the lemma must be satisfied.

It suffices to construct holomorphic eta-quotients in the remaining cases. Since $\Delta(z) \in M_{12}(\Gamma_0(N))$ for all N, this shows that there are always eta-quotients of weight 12.

Assume that $\epsilon_2(\Gamma_0(N)) = 0$. Then either $4|N$, in which case $\frac{\eta(z)^8}{\eta(2z)^4} \in M_2(\Gamma_0(N))$, or there is a prime $p|N$ with $p \equiv 3 \pmod 4$. In this case $\eta(z)^6\eta(pz)^6 \in M_6(\Gamma_0(N))$ and so there are eta-quotients of weight 6.

Assume that $\epsilon_3(\Gamma_0(N)) = 0$. Then either $9|N$, in which case $\frac{\eta(z)^6}{\eta(3z)^2} \in M_2(\Gamma_0(N))$, or there is a prime $p|N$ with $p \equiv 2 \pmod 3$. If $p > 2$ then $p \equiv 5 \pmod 6$ and $\eta(z)^4\eta(pz)^4 \in M_4(\Gamma_0(N))$. If $p = 2$ then $\frac{\eta(z)^{16}}{\eta(2z)^8} \in M_4(\Gamma_0(N))$. Thus, there is an eta-quotient of weight 4 when $\epsilon_3(\Gamma_0(N)) = 0$.

Assume now that $\epsilon_2(\Gamma_0(N)) = \epsilon_3(\Gamma_0(N)) = 0$. In the cases that $4|N$ or $9|N$ we have already constructed weight 2 eta-quotients. Otherwise, there is a prime $p|N$ with $p \equiv 3 \pmod 4$ and a prime $q|N$ with $q \equiv 2 \pmod 3$. If $p = q$, then $p \equiv 11 \pmod{12}$ and $\eta(z)^2\eta(pz)^2 \in M_2(\Gamma_0(N))$. If $p \neq q$ and $q > 2$ then $\eta(z)\eta(pz)\eta(qz)\eta(pqz) \in M_2(\Gamma_0(N))$, and if $p \neq q$ and $q = 2$ then $\frac{\eta(z)^4\eta(pz)^4}{\eta(2z)^2\eta(2pz)^2} \in M_2(\Gamma_0(N))$. Thus, there is an eta-quotient of weight 2 when $\epsilon_2(\Gamma_0(N)) = \epsilon_3(\Gamma_0(N)) = 0$. □

Now, we will prove Theorem 1.

Proof of Theorem 1. If the graded ring of modular forms for $\Gamma_0(N)$ is generated by eta-quotients, then $M_2(\Gamma_0(N))$ must be spanned by eta-quotients. Since $M_2(\Gamma_0(N))$ has positive dimension if $N > 1$, and there can be no weight 2 eta-quotients if $\Gamma_0(N)$ has elliptic points by Lemma 12, it follows that $\Gamma_0(N)$ has no elliptic points. (The only level 1 eta-quotients are powers of $\Delta(z)$, so the same conclusion follows when $N = 1$.)

In each case, we attempt to find enough eta-quotients to span $M_2(\Gamma_0(N))$. If we succeed and N is composite, then the fact that the graded ring of level N modular forms is generated in weight 2 proves the desired result. In cases where we fail to find enough eta-quotients, we enumerate all vectors in the corresponding $d(N) - 1$-dimensional lattice that correspond to weight 2 eta-quotients, and check that the dimension of the space they span is less than $\dim M_2(\Gamma_0(N))$ to prove that $M_2(\Gamma_0(N))$ is not generated by eta-quotients.

There are no prime levels $N \leq 500$ for which $M_2(\Gamma_0(N))$ is generated by eta-quotients. (In fact, there are no prime levels whatsoever, by Theorem 4.) Details about the computations are available at `http://users.wfu.edu/rouseja/eta/`. □

Before we prove Theorem 2, we need two lemmas first.

Lemma 13. *Suppose that $f(z)=\prod_{\delta|N}\eta(\delta z)^{r_\delta}$ is a level N eta quotient. Suppose that e is a positive integer and $\gcd(e,N)=1$. Let $r|e$ and view $f(z)|V(r)=\prod_{\delta|N}\eta(r\delta z)^{r_\delta}$ as an eta quotient of level eN. Let d be a divisor of eN and write $d=d_1d_2$, where $d_1|N$ and $d_2|e$. Then we have*

$$\mathrm{ord}_{\frac{1}{d}}(f(z)|V(r))=\frac{e\gcd(d_2^2,r^2)}{r\gcd(d_2^2,e)}\mathrm{ord}_{\frac{1}{d_1}}(f(z)).$$

Proof. We have $f(z)|V(r)=\prod_{\delta|eN}\eta(\delta z)^{s_\delta}$, where $s_\delta=r_{\delta/r}$ if $r|\delta$ and $\delta/r|N$. Otherwise $s_\delta=0$. The order of vanishing at the cusp $1/d$ is then

$$\frac{Ne}{24}\sum_{\substack{\delta|Ne\\ r|\delta}}\frac{\gcd(d,\delta)^2 s_\delta}{\gcd\left(d,\frac{Ne}{d}\right)d\delta}=\frac{Ne}{24}\sum_{\delta|N}\frac{\gcd(d,r\delta)^2 r_\delta}{\gcd\left(d,\frac{Ne}{d}\right)dr\delta}$$

$$=\frac{Ne}{24r}\sum_{\delta|N}\frac{\gcd(d_1,\delta)^2\gcd(d_2,r)^2 r_\delta}{\gcd\left(d_1,\frac{N}{d_1}\right)\gcd\left(d_2,\frac{e}{d_2}\right)d_1d_2\delta}=\frac{e\gcd(d_2^2,r^2)}{r\gcd(d_2^2,e)}\cdot\left(\frac{N}{24}\sum_{\delta|N}\frac{\gcd(d_1,\delta)^2 r_\delta}{\gcd\left(d_1,\frac{N}{d_1}\right)d_1\delta}\right)$$

$$=\frac{e\gcd(d_2^2,r^2)}{r\gcd(d_2^2,e)}\mathrm{ord}_{\frac{1}{d_1}}(f(z)).$$

□

The eta-quotients constructed in the next lemma will play a role in the proofs of many of the theorems.

Lemma 14. *If N is a positive integer, then for each divisor d of N, there is a holomorphic eta-quotient $E_{d,N}(z)\in M_{k_d}(\Gamma_0(N))$ that vanishes only at the cusps c/d. Moreover, if $E_{d,N}(z)=\prod_{\delta|N}\eta(\delta z)^{r_\delta}$, we have*

$$\frac{1}{2k_d}\sum_{\delta|N}|r_\delta|\leq\prod_{p|N}\left(\frac{p+1}{p-1}\right)^{\min\{2,\mathrm{ord}_p(N)\}}.$$

We follow the convention that an empty product on the right hand side takes the value 1.

Proof. We will prove our result by induction on the number of distinct prime factors of N. The base case is $N=1$. In this case we have $E_{1,1}(z)=\Delta(z)$ which vanishes at the only cusp of $\Gamma_0(1)$ and we have equality in the claimed inequality.

Assume now that N is a positive integer with k distinct prime factors, one of which is p and the highest power of p dividing N is p^m. By induction, for each divisor d of N/p^m, there is a form

$$F(z)=E_{d,N/p^m}(z)$$

that has all of its zeros at cusps with denominator d on $\Gamma_0(N/p^m)$.

Let $E_{d,N}(z) = \frac{F(z)^p}{F(z)|V(p)}$ and let d_1 and d_2 be divisors of N/p^m and p^m respectively. We now apply Lemma 13 with $e = p^m$, $r = 1$ or p. Then we get that

$$\operatorname{ord}_{1/d_1 d_2}(E_{d,N}(z)) = \left(\left(\frac{p^{m+1}}{\gcd(d_2^2, p^m)}\right) - \left(\frac{p^m \gcd(d_2^2, p^2)}{p \gcd(d_2^2, p^m)}\right)\right) \operatorname{ord}_{1/d_1}(E_{d,N}(z)).$$

It is clear that this quantity is always non-negative. Moreover, the expression in parentheses is equal to zero if $p|d_2$. If $d_1 \neq d$, then $\operatorname{ord}_{1/d_1}(E_{d,N}(z)) = 0$. This implies that all of the zeros of $E_{d,N}(z)$ occur at the cusps with denominator d, as desired. The passage from $F(z)$ to $E_{d,N}(z)$ multiplies the weight by $p-1$ and multiplies the sum of the absolute values of the exponents by $p+1$.

For $1 \leq s \leq m-1$, define

$$E_{dp^s,N}(z) = \frac{(F(z)|V(p^s))^{p^2+1}}{(F(z)|V(p^{s-1}))^p (F(z)|V(p^{s+1}))^p}.$$

Recall that d_1 is a divisor of N/p^m, d_2 is a divisor of p^m. We apply Lemma 13 with $e = p^m$, $r = p^{s-1}$, p^s or p^{s+1} and we get that

$$\begin{aligned}
&\operatorname{ord}_{1/d_1 d_2}(E_{dp^s,N}(z)) \\
&= \left(\frac{(p^2+1)\cdot p^m \gcd(d_2^2, p^{2s})}{p^s \gcd(d_2^2, p^m)} - \frac{p \cdot p^m \gcd(d_2^2, p^{2s-2})}{p^{s-1} \gcd(d_2^2, p^m)} - \frac{p \cdot p^m \gcd(d_2^2, p^{2s+2})}{p^{s+1} \gcd(d_2^2, p^m)}\right) \operatorname{ord}_{1/d_1}(F(z)) \\
&= p^m \left(\frac{(p^2+1) \gcd(d_2^2, p^{2s}) - p^2 \gcd(d_2^2, p^{2s-2}) - \gcd(d_2^2, p^{2s+2})}{p^s \gcd(d_2^2, p^m)}\right) \operatorname{ord}_{1/d_1}(F(z)).
\end{aligned}$$

If we write $d_2 = p^v$, the numerator of the above fraction is

$$p^{\min\{2v+2,2s+2\}} + p^{\min\{2v,2s\}} - p^{\min\{2v+2,2s\}} - p^{\min\{2v,2s+2\}}.$$

It is easy to see that if $s \neq v$, the above quantity is zero. If $s = v$, it is $p^{2s+2} - p^{2s}$. This shows that $E_{dp^s,N}(z)$ is non-vanishing unless $d_2 = p^s$ and $d_1 = d$. Moreover, the transfer from $F(z)$ to $E_{dp^s,N}(z)$ multiplies the weight by $(p-1)^2$ and the sum of the absolute values of the exponents by $(p+1)^2$. Hence, the stated inequality is true.

Finally, we let $E_{dp^m,N}(z) = \frac{(F(z)|V(p^m))^p}{F(z)|V(p^{m-1})}$. We have

$$\operatorname{ord}_{1/d_1 d_2}(E_{dp^m,N}(z)) = \left(\frac{p \cdot p^m \gcd(d_2^2, p^{2m})}{p^m \gcd(d_2^2, p^m)} - \frac{p^m \gcd(d_2^2, p^{2m-2})}{p^{m-1} \gcd(d_2^2, p^m)}\right) \operatorname{ord}_{1/d_1}(F(z)).$$

The quantity in parentheses is clearly zero unless $d_2 = p^m$. If $d_2 = p^m$, it is $p^{m+1} - p^{m-1}$. This shows that $E_{dp^m,N}(z)$ is non-vanishing except at cusps with denominator dp^m. Moreover, the transfer from $F(z)$ to $E_{dp^m,N}(z)$ multiplies the weight by $p-1$ and the sum of the absolute values of the exponents by $p+1$. This proves the desired result by induction on the number of distinct prime factors of N. □

Proof of Theorem 2. By Lemma 14, for each $d|N$, there is a form $E_{d,N}(z)$ that vanishes only at the cusps c/d. We may therefore write an arbitrary eta-quotient $f(z)$ in the form

$$f(z) = \prod_{d|N} E_{d,N}(z)^{r_d}$$

for a sequence of non-negative rational numbers r_d. Since the desired inequality is true for each $E_{d,N}$, it must be valid for $f(z)$ as well. □

Next, we prove Corollary 3

Proof of Corollary 3. Theorem 2 shows that if $p > 73$ and $f(z) = \prod_{\delta|4p} \eta(\delta z)^{r_\delta} \in M_2(\Gamma_0(4p))$, then $\sum_{\delta|4p} |r_\delta| \leq 36$ and $\sum_{\delta|4p} r_\delta = 4$. We explicitly enumerate all such exponents and we find that

$$\text{the number of eta-quotients } \in M_2(\Gamma_0(4p)) = \begin{cases} 10 & \text{if } p = 2 \\ 126 & \text{if } p = 3 \\ 76 & \text{if } p = 5 \\ 45 & \text{if } p = 7 \\ 28 & \text{if } p = 11 \\ 9 & \text{if } p \equiv 1 \pmod{24} \\ 16 & \text{if } p \equiv 5, 11, \text{ or } 17 \pmod{24} \text{ and } p > 11 \\ 21 & \text{if } p \equiv 7 \pmod{24} \text{ and } p > 7 \\ 15 & \text{if } p \equiv 13 \pmod{24} \\ 18 & \text{if } p \equiv 19 \pmod{24} \\ 37 & \text{if } p \equiv 23 \pmod{24}. \end{cases}$$

A straightforward calculation with this data and with the dimension formula for $M_2(\Gamma_0(4p))$ shows that if $p > 47$ then $\dim M_2(\Gamma_0(4p))$ is larger than the number of eta quotients it contains. Explicit calculations for $p \leq 73$ prove the desired result. □

Proof of Theorem 4. We will first prove that $\mathcal{E}_0(\Gamma_0(N))$ has finite codimension in $\mathcal{R}_0(\Gamma_0(N))$ when N is composite. Then N has at least three divisors, say 1, d and e with $1 < d < e$.

Let $E_{N,N}(z)$ be the eta-quotient from Lemma 14 that has zeros only at ∞, and let r be the weight of $E_{N,N}(z)$. Define

$$f_1(z) = \frac{\Delta(z)^{re-e+1}\Delta(dz)^{e-1}}{E_{N,N}(z)^{12e}} \text{ and } f_2(z) = \frac{\Delta(z)^{re-d+1}\Delta(ez)^{d-1}}{E_{N,N}(z)^{12e}}.$$

These are both modular functions for $\Gamma_0(N)$ and the orders of the poles of $f_1(z)$ and $f_2(z)$ at infinity are both

$$M = 12e\text{ord}_\infty(E_{N,N}) - (ke + de - d - e + 1).$$

Since $\Delta(z) = q - 24q^2 + O(q^3)$, it follows that $\Delta(z)^s = q^s - 24sq^{s+1} + O(q^{s+2})$, and since the power of $\Delta(z)$ in $f_1(z)$ and $f_2(z)$ are different, the coefficients of q^{-M+1} in $f_1(z)$ and $f_2(z)$ are different.

It follows that $\mathcal{E}_0(\Gamma_0(N))$ contains $f_1(z)$, which has a pole of order M at infinity, and $f_1(z) - f_2(z)$, which has a pole of order $M-1$ at infinity. Since every positive integer $n \geq M^2 - 3M + 1$ can be written as a non-negative linear combination of M and $M-1$, the ring $\mathcal{E}_0(\Gamma_0(N))$ contains a function with pole of order n at infinity for every $n \geq M^2 - 3M + 1$. It follows that every element of $\mathcal{R}_0(\Gamma_0(N))$ can be written in the form

$$f(z) = s(z) + t(z)$$

where $s(z) \in \mathcal{E}_0(\Gamma_0(N))$ and $\text{ord}_\infty t(z) \geq -(M^2 - 3M + 1)$. The set of functions $t(z) \in \mathcal{R}_0(\Gamma_0(N))$ with $\text{ord}_\infty t(z) \leq -(M^2 + 3M + 1)$ is a finite dimensional vector space (of dimension at most $M^2 - 3M + 1$) and this proves the result. Since $\mathcal{E}_k(\Gamma_0(N))$ is a module for $\mathcal{E}_0(\Gamma_0(N))$, we have that $\mathcal{E}_k(\Gamma_0(N))$ has finite codimension in $\mathcal{R}_k(\Gamma_0(N))$ if $\mathcal{E}_k(\Gamma_0(N))$ is nonempty. This occurs precisely when the conditions of Lemma 12 are satisfied.

Suppose now that $N = p$ is prime. Any element in $\mathcal{E}_k(\Gamma_0(p))$ is a linear combination of those of the form $\eta(z)^a \eta(pz)^{2k-a}$ for some even integer a. The order of vanishing of this form at infinity is $\frac{2kp - a(p-1)}{24}$. We see that if a and a' are both such that $f_1(z) = \eta(z)^a \eta(pz)^{2k-a}$ and $f_2(z) = \eta(z)^{a'} \eta(pz)^{2k-a'}$ are level p eta-quotients, then the orders of vanishing of $f_1(z)$ and $f_2(z)$ at infinity must be congruent modulo $r = \frac{p-1}{\gcd(24, p-1)}$.

However, the Riemann-Roch theorem implies that if g is the genus of $X_0(N)$ and $m \geq 2g$, then there is an element in $\mathcal{R}_0(\Gamma_0(p))$ with a pole of order m at infinity. The fact that $\mathcal{R}_k(\Gamma_0(p))$ is non-empty and is a module for $\mathcal{R}_0(\Gamma_0(p))$ implies that every sufficiently large positive integer occurs as the order of pole of an element of $\mathcal{R}_k(\Gamma_0(p))$. This easily implies that if

$$r = \frac{p-1}{\gcd(24, p-1)} > 1$$

then $\mathcal{E}_k(\Gamma_0(N))$ does not have finite codimension in $\mathcal{R}_k(\Gamma_0(N))$. We have that $\frac{p-1}{\gcd(24,p-1)} = 1$ if and only if $p - 1 | 24$ and this occurs precisely for $p = 2, 3, 5, 7$ and 13. If $p = 2, 3, 5, 7$ or 13, the fact that the order of vanishing of $f(z)$ at ∞ is -1 implies that $\mathcal{E}_0(\Gamma_0(p)) = \mathcal{R}_0(\Gamma_0(p))$. This proves the desired result. □

Proof of Theorem 5. Again, let $E_{N,N}(z)$ be the holomorphic eta-quotient with zeros only at the cusp at ∞ from Lemma 14. Let the weight of $E_{N,N}$ be r.

First, we will show that (1) $\implies$ (2). Suppose that for all positive integers k, every element of $M_k(\Gamma_0(N))$ is a linear combination of weakly holomorphic eta-quotients with poles only at infinity. Then there must be weight 2 weakly holomorphic eta-quotients which implies (by Lemma 12) that $\Gamma_0(N)$ has no elliptic points. Suppose that $f \in \mathcal{R}_k(\Gamma_0(N))$. There is a positive integer m for which $f E_{N,N}^m$ is holomorphic at infinity, and since f only has poles at infinity,

$fE_{N,N}^m \in M_{k+mr}(\Gamma_0(N))$. By assumption, therefore, we may write

$$fE_{N,N}^m = \sum_i c_i g_i$$

where the g_i are weakly holomorphic eta-quotients with poles only at infinity and so

$$f = \sum_i c_i \frac{g_i}{E_{N,N}^m} \in \mathcal{E}_k(\Gamma_0(N))$$

is a linear combination of weakly holomorphic eta-quotients with poles only at infinity, and so $\mathcal{R}_k(\Gamma_0(N)) = \mathcal{E}_k(\Gamma_0(N))$ for all positive even k. Observe that the map $T : \mathcal{R}_k(\Gamma_0(N)) \to \mathcal{R}_{r+k}(\Gamma_0(N))$ given by $T(f) = fE_{N,N}$ is an isomorphism. Similarly, the map $T : \mathcal{E}_k(\Gamma_0(N)) \to \mathcal{E}_{r+k}(\Gamma_0(N))$ given by $T(f) = fE_{N,N}$ is also an isomorphism, and so $\mathcal{R}_k(\Gamma_0(N)) = \mathcal{E}_k(\Gamma_0(N))$ if and only if $\mathcal{R}_{k+r}(\Gamma_0(N)) = \mathcal{E}_{k+r}(\Gamma_0(N))$. Hence, since $\mathcal{R}_r(\Gamma_0(N)) = \mathcal{E}_r(\Gamma_0(N))$, it follows that $\mathcal{R}_0(\Gamma_0(N)) = \mathcal{E}_0(\Gamma_0(N))$. Thus, (1) $\implies$ (2).

Note that (2) $\implies$ (1) is trivial, since if $f(z) \in M_k(\Gamma_0(N))$, then $f(z) \in \mathcal{R}_k(\Gamma_0(N)) = \mathcal{E}_k(\Gamma_0(N))$, and so $f(z)$ is a linear combination of eta-quotients with poles only at ∞.

It is obvious that (2) $\implies$ (3). We will now prove that (3) $\implies$ (1).

For all $N \geq 1$, $\mathcal{R}_2(\Gamma_0(N))$ is non-zero, since if $N > 1$ it contains $M_2(\Gamma_0(N))$, which has positive dimension, and for $N = 1$, it contains $\frac{E_4(z)^2 E_6(z)}{\Delta(z)}$. Therefore, the hypothesis that $\mathcal{R}_2(\Gamma_0(N)) = \mathcal{E}_2(\Gamma_0(N))$ implies that $\mathcal{E}_2(\Gamma_0(N))$ is non-empty. Lemma 12 implies that $\Gamma_0(N)$ has no elliptic points. In particular, N cannot equal 2, 3, 5, 7 or 13.

Also, the hypothesis that $\mathcal{E}_2(\Gamma_0(N)) = \mathcal{R}_2(\Gamma_0(N))$ implies (by Theorem 4) that $N \neq 11$ and N cannot be a prime greater than 13. Thus, N is composite.

Since N is composite and $\Gamma_0(N)$ has no elliptic points, the graded ring of modular forms for $\Gamma_0(N)$ is generated in weight 2. Since $M_2(\Gamma_0(N)) \subseteq \mathcal{R}_2(\Gamma_0(N)) = \mathcal{E}_2(\Gamma_0(N))$, every weight 2 modular form is a linear combination of weakly holomorphic eta-quotients with poles only at infinity. Thus, the graded ring of modular forms is generated by forms that are linear combinations of weakly holomorphic eta-quotients, and this proves that (3) $\implies$ (1). □

Proof of Theorem 6. Again, let $E_{N,N}$ be the level N eta-quotient of weight r with zeros only at infinity from Lemma 14. In each case, we are able to prove that every form in $M_{r+2}(\Gamma_0(N))$ is a linear combination of (holomorphic) eta-quotients and this proves that every element of $M_2(\Gamma_0(N))$ is a linear combination of weakly holomorphic eta-quotients. Since the graded ring of modular forms of level N is generated in weight 2, this proves that every element of $M_k(\Gamma_0(N))$ is a linear combination of weakly holomorphic eta-quotients. Then, by Theorem 5, we have that $\mathcal{R}_2(\Gamma_0(N)) = \mathcal{E}_2(\Gamma_0(N))$. For code to compute eta-quotients in $M_{r+2}(\Gamma_0(N))$, see `http://users.wfu.edu/rouseja/eta/`. □

Before we begin the proof of Theorem 7, we will need a series of preliminary results.

Lemma 15. *Let $g(z) \in M_k(\Gamma_0(N)) \cap \mathbb{Q}((q))$ have the property that g is non-zero on the upper half plane and let $d|N$ with $d > 0$. Then for any $\ell \in \mathbb{Z}$ with $\gcd(\ell, d) = 1$, we have $\mathrm{ord}_{\frac{1}{d}}(g(z)) = \mathrm{ord}_{\frac{\ell}{d}}(g(z))$.*

Proof. Since $k \in 2\mathbb{Z}$, there exist integers a, b such that $4a + 6b = k$. This implies that

$$G(z) := \frac{g(z)}{E_4(z)^a E_6(z)^b}$$

is a meromorphic modular function for $X_0(N)$. Since $E_4(z), E_6(z) \in \mathbb{Z}[[q]]$, $G(z) \in \mathbb{Q}((q))$ as well. Thus for any $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have $G(z)^\tau = G(z)$, and thus $\mathrm{div}(G(z))^\tau = \mathrm{div}(G(z))$. Furthermore, at any cusp $\frac{\ell}{d} \in \mathbb{Q}$, we have $\mathrm{ord}_{\frac{\ell}{d}}(G(z)) = \mathrm{ord}_{\frac{\ell}{d}}(g(z))$. We note that since $G(z)$ is a meromorphic function on $\Gamma_0(N)$ with rational coefficients, we have $G(z) \in \mathbb{Q}(j(z), j(Nz))$ where j is the modular j-function.

We note that $\mathbb{Q}(j(z), j(Nz))$ is the function field for a model of the modular curve $X_0(N)/\mathbb{Q}$. For $d, \ell \in \mathbb{Z}$ with $\gcd(\ell, d) = 1$, we let $\left[\begin{smallmatrix} \ell \\ d \end{smallmatrix}\right]$ denote the point on $X_0(N)$ associated to the cusp $\frac{\ell}{d}$. We will use the following result.

Theorem 16 (adapted from Theorem 1.3.1 in [17])**.**
(1) *The cusps of $X_0(N)$ are rational over $\mathbb{Q}(\zeta_N)$, where $\zeta_N = e^{2\pi i/N}$.*
(2) *For $s \in (\mathbb{Z}/N\mathbb{Z})^\times$ let $\tau_s \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ be defined by*

$$\tau_s : \zeta_N \mapsto \zeta_N^s .$$

Then

$$\begin{bmatrix} \ell \\ d \end{bmatrix}^{\tau_s} = \begin{bmatrix} \ell \\ s'd \end{bmatrix}$$

where $s' \in \mathbb{Z}$ is chosen so that $ss' \equiv 1 \pmod{N}$.

Since $\mathrm{div}(G(z))$ is fixed by each element of $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, part 1 of Theorem 16 implies that for any two cusps $\frac{j}{d}, \frac{j'}{d'}$ such that there exists some $\tau_s \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ with $\left[\begin{smallmatrix} j \\ d \end{smallmatrix}\right]^{\tau_s} = \left[\begin{smallmatrix} j' \\ d' \end{smallmatrix}\right]$, we have $\mathrm{ord}_{\frac{j}{d}}(G(z)) = \mathrm{ord}_{\frac{j'}{d'}}(G(z))$. We will finish the proof of the lemma by showing that if $d|N$ and $\ell \in \mathbb{Z}$ with $\gcd(d, \ell) = 1$, then

$$\begin{bmatrix} 1 \\ d \end{bmatrix}^{\tau_{\ell^*}} = \begin{bmatrix} \ell \\ d \end{bmatrix} \tag{3}$$

where $\ell^* \in \mathbb{Z}$ with $\ell^* \equiv \ell \pmod{d}$ and $\gcd(\ell^*, N) = 1$.

Choose ℓ' such that $\ell^*\ell' \equiv 1 \pmod{N}$. Then there exists some $t \in \mathbb{Z}$ such that $\ell^*\ell' + tN = 1$, and $A, B \in \mathbb{Z}$ such that $A\ell' - BdtN = 1$. Consider the matrix

$$\gamma := \begin{bmatrix} A & B \\ dtN & \ell' \end{bmatrix} \in \Gamma_0(N) .$$

One computes that

$$\gamma\left(\frac{1}{\ell^* d}\right) = \frac{A + B\ell^* d}{d}\,.$$

Our choice of A implies that $A \equiv \ell \pmod{d}$. Thus there exists some $r \in \mathbb{Z}$ such that

$$\frac{A + B\ell^* d}{d} = \frac{\ell}{d} + r\,.$$

This implies that $\left[\begin{smallmatrix}\ell \\ d\end{smallmatrix}\right] = \left[\begin{smallmatrix}1 \\ \ell^* d\end{smallmatrix}\right]$, which by part 2 of Theorem 16 implies (3), finishing the proof of the lemma. □

Lemma 17. *Suppose $f(z) \in M_k(\Gamma_0(N)) \cap \mathbb{Z}[[q]]$ has the property that $f(z)$ is non-zero on the upper half plane. Then there exists a positive integer c such that $f(z)^c = ag(z)$ where $a \in \mathbb{Z}$ and $g(z)$ is an eta-quotient.*

Proof. Let $d, j \in \mathbb{Z}$ with $d|N$ and $\gcd(j, n) = 1$. Then by Lemma 15, we have $\mathrm{ord}_{\frac{1}{d}}(f(z)) = \mathrm{ord}_{\frac{j}{d}}(f(z))$ for all such d and j. For each divisor d of N, define

$$c_d = \begin{cases} \dfrac{\mathrm{lcm}(\mathrm{ord}_{\frac{1}{d}}(f(z)), \mathrm{ord}_{\frac{1}{d}}(E_{d,N}(z)))}{\mathrm{ord}_{\frac{1}{d}}(f(z))} & \text{if } \mathrm{ord}_{\frac{1}{d}}(f(z)) \neq 0\,, \\ 1 & \text{if } \mathrm{ord}_{\frac{1}{d}}(f(z)) = 0\,, \end{cases}$$

and

$$r_d = \begin{cases} \dfrac{\mathrm{lcm}(\mathrm{ord}_{\frac{1}{d}}(f(z)), \mathrm{ord}_{\frac{1}{d}}(E_{d,N}(z)))}{\mathrm{ord}_{\frac{1}{d}}(E_{d,N}(z))} & \text{if } \mathrm{ord}_{\frac{1}{d}}(f(z)) \neq 0\,, \\ 0 & \text{if } \mathrm{ord}_{\frac{1}{d}}(f(z)) = 0\,, \end{cases}$$

where $E_{d,N}$ is defined as in Lemma 14. Let $c = \prod c_d$. Now define

$$F(z) = \prod_{d|N} E_{d,N}(z)^{r_d c/c_d}\,.$$

One can check using Lemma 15 and the definition of $E_{d,N}(z)$ that at each cusp $\frac{j}{d}$, we have

$$\mathrm{ord}_{\frac{j}{d}}(f(z)^c) = \mathrm{ord}_{\frac{j}{d}}(F(z)) = \begin{cases} \dfrac{c}{c_d}\,\mathrm{lcm}(\mathrm{ord}_{\frac{1}{d}}(f(z)), \mathrm{ord}_{\frac{1}{d}}(E_{d,N}(z))) & \text{if } \mathrm{ord}_{\frac{1}{d}}(f(z)) \neq 0\,, \\ 0 & \text{if } \mathrm{ord}_{\frac{1}{d}}(f(z)) = 0\,. \end{cases}$$

Since $f(z)^c$ and $F(z)$ are both holomorphic and non-zero on the upper-half plane, this implies that $f(z)^c/F(z) \in M_0^!(\Gamma_0(N)) \cap \mathbb{Z}((q))$ is a holomorphic non-zero function on the upper-half plane and at the cusps. We conclude that there exists some $A \in \mathbb{C}$ such that $f(z)^c = AF(z)$, which completes the proof of the lemma. □

Corollary 18. *Suppose $f(z) = \sum_{n=n_0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N)) \cap \mathbb{Z}[[q]]$ has the property that $f(z)$ is non-zero on the upper half plane. Then $\frac{1}{a(n_0)} f(z) \in \mathbb{Z}[[q]]$, that is, every coefficient of $f(z)$ is divisible by the first non-zero coefficient.*

Proof. Using the same notation as in the proof of Lemma 17, since the leading non-zero coefficient of the Fourier expansion of $F(z)$ is one, we have that $A = a(n_0)^c \in \mathbb{Z}$. If $|a(n_0)| = 1$, the statement of the lemma is trivially true. Now suppose $|a(n_0)| > 1$, and let $p|a(n_0)$ be prime. Let $f(z) = \sum_{n=n_0}^{\infty} a(n)q^n$ and define two series $S(q), R(q) \in \mathbb{Z}[[q]]$ by

$$S(q) = \sum_{n=n_0}^{\infty} \left\lfloor \frac{a(n)}{p} \right\rfloor q^n \,,$$

$$R(q) = \sum_{n=n_0}^{\infty} \left(a(n) - p \left\lfloor \frac{a(n)}{p} \right\rfloor \right) q^n \,.$$

Note that each coefficient of $R(q)$ is in the set $[0, p-1] \cap \mathbb{Z}$. Since $f(z) = pS(q) + R(q)$,

$$\frac{1}{p} f(z)^c = \sum_{t=0}^{c} p^{t-1} \binom{c}{t} S(q)^t R(q)^{c-t} = \frac{a(n_0)^c}{p} F(z) \,.$$

For each $t \geq 1$, each summand above lies in $\mathbb{Z}[[q]]$. Since $p|a(n_0)$ and $F(z) \in \mathbb{Z}[[q]]$, we see that $\frac{1}{p} f(z)^c \in \mathbb{Z}[[q]]$ as well. This implies that the summand $\frac{1}{p} R(q)^c \in \mathbb{Z}[[q]]$. Suppose $R(q) \neq 0$ and let $1 \leq b(n') \leq p-1$ be the first non-zero coefficient of $R(q)$. Then the cn'-th coefficient of $\frac{1}{p} R(q)^c$ would be $\frac{b(n')^c}{p}$ which would imply that $p|b(n')$, contradicting that $R(q)$ is non-zero. Thus $\frac{1}{p} f(z) \in \mathbb{Z}[[q]]$. Iterating this argument over the prime factorization of $a(n_0)$ leads to the desired conclusion: $\frac{1}{a(n_0)} f(z) \in \mathbb{Z}[[q]]$. ☐

Lemma 19. *Suppose $h(z) \in \mathbb{Z}[[q]]$ can be expressed as $h(z) = \prod_{d|N} \eta(dz)^{s_d}$ where each $s_d \in \mathbb{Q}$. Then each $s_d \in \mathbb{Z}$–in other words, $h(z)$ is an eta-quotient.*

Proof. Let $d_0 = 1, d_1, d_2, \ldots, d_t = N$ be the divisors of N, and let $r = \mathrm{ord}_\infty(h(z))$. Newton's generalized binomial theorem implies

$$\prod_{n \geq 1} (1 - q^{dn})^s = 1 - sq^d + O(q^{2d}) \,.$$

Thus we have

$$h(z) = q^r \prod_{m=0}^{t} (1 - s_{d_m} q^{d_m} + \ldots) \,.$$

We proceed by induction on the divisors of N. For $d_0 = 1$, we see that $-s_{d_0}$ is the coefficient of q^{r+1} in the Fourier expansion of $h(z)$. Since all of $h(z)$'s coefficients are integral, we conclude that $s_{d_0} \in \mathbb{Z}$. Now suppose that $s_{d_m} \in \mathbb{Z}$ for all $0 \leq m < \ell$. Write

$$q^r \prod_{m=0}^{\ell-1} \prod_{n \geq 1} (1 - q^{d_m n})^{s_{d_m}} = \sum_{n=r}^{\infty} b(n) q^n \,,$$

where each $b(n) \in \mathbb{Z}$. Then the coefficient of q^{r+d_ℓ} in the Fourier expansion of $h(z)$ would be $b(r + d_\ell) - s_{d_\ell}$, and again we conclude that $s_{d_\ell} \in \mathbb{Z}$ as well. ☐

Proof of Theorem 7. By Lemma 17 and Corollary 18, $f(z) = \sum_{n=n_0}^{\infty} a(n)q^n$ is an integer multiple of a quotient of eta-functions to rational powers with $\frac{1}{a(n_0)} f(z) \in \mathbb{Z}[[q]]$. Lemma 19 then implies that $f(z)$ is in fact an eta-quotient multiplied by $a(n_0)$. □

We now turn to the proof of Theorem 10. When $1 \leq k \leq 6$, the theorem can be verified by direct computations. Suppose $k \geq 7$. Our first step will be to identify a minimal generating set for the group $\mathcal{M}(2^k)$. Define the set $\{f_{\ell,k}(z)\}_{\ell=0}^{k-1} \subset \mathcal{M}(2^k)$ as follows. Let

$$f_{0,k}(z) = \frac{\Delta(2^{k-1}z)}{\Delta(2^k z)}\,.$$

For $1 \leq \ell \leq 3$, let

$$f_{\ell,k}(z) = \begin{cases} \dfrac{\eta(2^\ell z)^5 \eta(2^{k-1}z)^4}{\eta(2^{\ell-1}z)^2 \eta(2^{\ell+1}z)^2 \eta(2^{k-2}z)\eta(2^k z)^4} & \text{if } \ell \equiv k \pmod 2\,, \\ \dfrac{\eta(2^\ell z)^5 \eta(2^{k-1}z)}{\eta(2^{\ell-1}z)^2 \eta(2^{\ell+1}z)^2 \eta(2^k z)^2} & \text{if } \ell \not\equiv k \pmod 2\,. \end{cases}$$

For $4 \leq \ell \leq k-2$, let

$$f_{\ell,k}(z) = \begin{cases} \dfrac{\eta(2^\ell z)^2 \eta(2^{k-1}z)}{\eta(2^{\ell-1}z)\eta(2^k z)^2} & \text{if } \ell \equiv k \pmod 2\,, \\ \dfrac{\eta(2^\ell z)^2 \eta(2^{k-1}z)^4}{\eta(2^{\ell-1}z)\eta(2^{k-2}z)\eta(2^k z)^4} & \text{if } \ell \not\equiv k \pmod 2\,. \end{cases}$$

Finally, let

$$f_{k-1,k}(z) = \frac{\eta(2^{k-1}z)^6}{\eta(2^{k-2}z)^2 \eta(2^k z)^4}\,.$$

Lemma 20. *The set $\{f_{\ell,k}\}_{\ell=0}^{k-1}$ is a minimal generating set for the multiplicative group $\mathcal{M}(2^k)$.*

Proof. Let $g(z) = \prod_{i=0}^{k} \eta(2^i z)^{r_i} \in \mathcal{M}(2^k)$. Let i^* be the minimal i such that r_i is non-zero. If $i^* \leq 2$, then r_{i^*} must be even since $\sum_{i=i^*}^{k} 2^i r_i \equiv 0 \pmod{2^{i^*+1}}$ by (2). Suppose that $i^* = k-2$. Then $r_{k-2} + r_{k-1} + r_k = 0$ since $g(z)$ is weight 0. By (2), the congruence

$$4r_{k-2} + 2r_{k-1} + r_k \equiv 0 \pmod{24}$$

must hold, which implies that r_k must be even. If k is odd, then $r_{k-2} + r_k \equiv 0 \pmod 2$ by the third condition in (2). Thus r_{k-2} must be even as well. On the other hand, if k is even, then by the third condition in (2), r_{k-1} must be even. Plugging in $-r_{k-2} - r_{k-1}$ into the congruence above leads to $r_{k-1} \equiv -3r_{k-2} \pmod{24}$. This implies that $r_{k-2} \equiv r_{k-1} \pmod 2$, and r_{k-2} is even as well. If $i^* = k-1$, then $r_{k-1} = -r_k$ since $g(z)$ is weight 0. Combining this with the congruence condition $2r_{k-1} + r_k \equiv 0 \pmod{24}$ implies that r_{k-1} is a multiple of 24, and thus $g(z)$ is an integer power of $f_{0,k}(z)$.

The proof of Lemma 20 now follows in this way. For $g(z) \in \mathcal{M}(2^k)$, if $i^* < k-1$, then r_{i^*} is a multiple of the power of $\eta(2^{i^*} z)$ in $f_{i^*+1,k}(z)$. Thus for the appropriate power c, $f_{i^*+1,k}(z)^c g(z) = \prod_{i=i^*+1}^{k} \eta(2^i z)^{r'_i} \in \mathcal{M}(2^k)$. This process continues until we are left with a function which is an integer power of $f_{0,k}(z)$. □

Proof of Theorem 10. We continue our assumption that $k \geq 7$, and make the further restriction that k be even. The proof for odd k is very similar.

For odd j, let the cusp $\frac{j}{2^t} \in \mathbb{Q}$ correspond to the point $\left[\frac{j}{2^t}\right] \in X_0(2^k)$. Define $C := \left\{\left[\frac{j}{2^t}\right]\right\}$ to be a complete set of representatives of the cusps of $X_0(2^k)$. For any divisor $d = \sum_{\left[\frac{j}{2^t}\right] \in C} n_{j,t} \left[\frac{j}{2^t}\right] \in \mathrm{Div}^0(X_0(2^k)/\mathbb{Q})$ whose support only includes cusps, it follows from the proof of Lemma 15 that $n_{j,t} = n_{j',t}$ for all cusps $[\frac{j}{2^t}], [\frac{j'}{2^t}]$ of $X_0(2^k)$ that share a common denominator. For $1 \leq \ell \leq k-1$, $\ell \neq \frac{k}{2}$, define the divisors $d_{\ell,k} = \sum_{\left[\frac{j}{2^t}\right] \in C} n_{t,\ell,k} \left[\frac{j}{2^t}\right]$ as follows. For $\ell = 1$ let

$$n_{t,1,k} = \begin{cases} 1 & \text{for } t = 1 \\ 0 & \text{for } t = 0, 2 \leq t \leq k-1 \\ -1 & \text{for } t = k\,. \end{cases}$$

For $\ell = 2$ let

$$n_{t,2,k} = \begin{cases} 1 & \text{for } t = 2, k-1 \\ 0 & \text{for } t = 0, 1\,,\ 3 \leq t \leq k-2 \\ -3 & \text{for } t = k\,. \end{cases}$$

For $\ell = 3$ let

$$n_{t,3,k} = \begin{cases} 1 & \text{for } t = 3 \\ 0 & \text{for } t = 0, 1, 2\,,\ 4 \leq t \leq k-1 \\ -4 & \text{for } t = k\,. \end{cases}$$

For $4 \leq \ell < \frac{k}{2}$, ℓ odd, let

$$n_{t,\ell,k} = \begin{cases} 1 & \text{for } t = \ell \\ 0 & \text{for } 0 \leq t \leq \ell-1\,,\ \ell+1 \leq t \leq k-2 \\ 2^{\ell-1} & \text{for } t = k-1 \\ -2^{\ell} & \text{for } t = k\,. \end{cases}$$

For $4 \leq \ell < \frac{k}{2}$, ℓ even, let

$$n_{t,\ell,k} = \begin{cases} 1 & \text{for } t = \ell \\ 0 & \text{for } 0 \leq t \leq \ell-1\,,\ \ell+1 \leq t \leq k-2 \\ -2^{\ell-2} & \text{for } t = k-1 \\ -2^{\ell-2} & \text{for } t = k\,. \end{cases}$$

For $\frac{k}{2} < \ell \leq k-3$, ℓ odd, let

$$n_{t,\ell,k} = \begin{cases} 0 & \text{for } 0 \leq t \leq \ell - 1 \\ 1 & \text{for } \ell \leq t \leq k-2 \\ 1 + 2^{k-\ell-1} & \text{for } t = k-1 \\ -3 \cdot 2^{k-\ell-1} + 1 & \text{for } t = k\,. \end{cases}$$

For $\frac{k}{2} < \ell \leq k-2$, ℓ even, let

$$n_{t,\ell,k} = \begin{cases} 0 & \text{for } 0 \leq t \leq \ell - 1 \\ 1 & \text{for } \ell \leq t \leq k-1 \\ -2^{k-\ell} + 1 & \text{for } t = k\,. \end{cases}$$

For $\ell = k-1$, let

$$n_{t,\ell,k} = \begin{cases} 0 & \text{for } 0 \leq t \leq k-2 \\ 1 & \text{for } t = k-1 \\ -1 & \text{for } t = k\,. \end{cases}$$

One calculates that for $1 \leq \ell \leq 3$,

$$2^{k-\ell-3} d_{\ell,k} = 2^{i_{k-2\ell-1,k}} d_{\ell,k} = \operatorname{div}(f_{\ell,k})\,, \tag{4}$$

for $4 \leq \ell < \frac{k}{2}$,

$$2^{k-\ell-3} d_{\ell,k} = 2^{i_{k-2\ell-1,k}} d_{\ell,k} = 2\operatorname{div}(f_{\ell,k}) - \operatorname{div}(f_{\ell+1,k})\,, \tag{5}$$

for $\frac{k}{2} < \ell \leq k-2$,

$$2^{\ell-4} d_{\ell,k} = 2^{i_{2\ell-k-2,k}} d_{\ell,k} = \operatorname{div}(f_{\ell,k})\,, \tag{6}$$

and for $\ell = k-1$,

$$2^{k-4} d_{k-1,k} = 2^{i_{k-2,k}} d_{\ell,k} = \operatorname{div}(f_{k-1,k})\,. \tag{7}$$

We note that the constants in front of the divisors $d_{\ell,k}$ in (4) to (7) match the orders of the cyclic subspaces in the statement of the theorem. Thus to prove the statement, it suffices to show that the $d_{\ell,k}$'s generate all of $J_0(2^k)(\mathbb{Q})_{\text{cusp}}$, and that each cyclic subspace $\langle d_{\ell,k}\rangle$'s intersection with the subspace $\langle d_{1,k}, \ldots, d_{\ell-1,k}, d_{\ell+1,k}, \ldots\rangle$ is trivial. Since $f_{0,k} = \Delta(2^{k-1}z)/\Delta(2^k z)$ has a single zero at the cusp 0, without loss of generality it can be assumed that each divisor in $J_0(2^k)(\mathbb{Q})_{\text{cusp}}$ is not supported at the cusp $[0]$. Now to show that the $d_{\ell,k}$'s generate all of $J_0(2^k)(\mathbb{Q})_{\text{cusp}}$, it remains to show that there exists a divisor $s = \sum_{\left[\frac{j}{2^t}\right]\in C} n_{t,s} \left[\frac{j}{2^t}\right]$ where $n_{t,s} = 0$ for $0 \leq t < \frac{k}{2}$ and $n_{\frac{k}{2},s} = 1$. One such divisor is

$$s = \left(\sum_{4\leq\ell<\frac{k}{2}} 2^{k-2\ell} d_{\ell,k}\right) - \operatorname{div}(f_{4,k})\,.$$

We now show that each cyclic subspace $\langle d_{\ell,k}\rangle$'s intersection with $\langle d_{1,k},\ldots,d_{\ell-1,k},d_{\ell+1,k},\ldots\rangle$ is trivial. Examining the divisors in $\langle d_{\ell,k}\rangle$, when $1\le\ell\le 3$ and $\frac{k}{2}<\ell\le k-1$, we note that the support for each multiple of $d_{\ell,k}$ does not include the cusps $\left[\frac{j}{2^t}\right]$ for $t<\ell$ but does include the cusps $\left[\frac{j}{2^\ell}\right]$. For this reason they cannot be linear combinations of the other generators. At this point, if there exists some non-trivial relation between the $d_{\ell,k}$'s, it must involve only the ℓ's with $4\le\ell<k/2$. Since this interval is empty if $k=8$, we let $k\ge 10$. Suppose $\sum_{\ell=4}^{k/2-1} c_\ell d_{\ell,k}=\operatorname{div}(g)$ for some $g\in\mathcal{M}(2^k)$. Because $d_{\ell,k}$ has order $2^{k-\ell-3}$, we may assume that either $c_\ell=0$, or $0\le\operatorname{ord}_2(c_\ell)<k-\ell-3$. Utilizing the relations in (4) to (7), we see that $g=\prod_{\ell=4}^{k/2-1}(f_{\ell,k}^2/f_{\ell+1,k})^{c_\ell/2^{k-\ell-3}}$. When we write $g=\prod\eta(2^tz)^{r_t}$, we see for $3\le t\le k/2$ that

$$r_t=\begin{cases}-2\frac{c_4}{2^{k-7}} & t=3\\ 5\frac{c_4}{2^{k-7}}-2\frac{c_5}{2^{k-8}} & t=4\\ -2\frac{c_{t-1}}{2^{k-t-2}}+5\frac{c_t}{2^{k-t-3}}-2\frac{c_{t+1}}{2^{k-t-4}} & 5\le t\le\frac{k}{2}-2\\ -2\frac{c_{\frac{k}{2}-2}}{2^{\frac{k}{2}-1}}+5\frac{c_{\frac{k}{2}-1}}{2^{\frac{k}{2}-2}} & t=\frac{k}{2}-1\\ -2\frac{c_{\frac{k}{2}-1}}{2^{\frac{k}{2}-2}} & t=\frac{k}{2}\,.\end{cases}$$

Let t^* be the minimal t in $4\le t<\frac{k}{2}-1$ with $c_{t^*}\ne 0$. Since each $r_t\in\mathbb{Z}$ by Lemma 19, examining r_{t^*-1} and using $0\le\operatorname{ord}_2(c_{t^*})<k-t^*-3$, we see that $\operatorname{ord}_2(c_{t^*})=k-t^*-4$ which then implies that $\operatorname{ord}_2(c_{t^*+1})=k-t^*-6$ since $r_{t^*}\in\mathbb{Z}$. Repeating this argument for each successive r_t with $t^*\le t<\frac{k}{2}$, we see that $\operatorname{ord}_2(c_{t+1})=k+t^*-6-2t$. However, this would imply that $\operatorname{ord}_2(r_{k/2})=t^*-\frac{k}{2}$, which contradicts that $r_{\frac{k}{2}}\in\mathbb{Z}$. □

Proof of Theorem 11. We will show that there is a bijection between weight k eta-quotients in $M_k(\Gamma_0(N))$ and cuspidal divisors of degree $\frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]$ with non-negative multiplicity at each cusp. Since there are $\phi\left(\gcd\left(d,\frac{N}{d}\right)\right)$ cusps with denominator d, the claimed result follows.

It is well-known that if $f(z)\in M_k(\Gamma_0(N))$, then the number of zeros of $f(z)$ on a fundamental domain for $\Gamma_0(N)$ is equal to $\frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]$. (This follows from Proposition 2.16 of [16] together with an application of the Riemann-Hurwitz formula for the genus of $X_0(N)$ derived from the covering $X_0(N)\to X_0(1)$.)

If $f(z)\in M_k(\Gamma_0(N))$ is an eta-quotient, let $c_d=\operatorname{ord}_{1/d}(f(z))$. It follows that

$$\sum_{d|N} c_d\phi\left(\gcd\left(d,\frac{N}{d}\right)\right)=\frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)],$$

and so $\operatorname{div}(f)$ is a cuspidal divisor of the appropriate degree.

Suppose now $X_0(N)$ has genus zero, $h(z)\in M_k(\Gamma_0(N))$ is an eta-quotient, and that $(c_d:d|N)$ is a sequence of non-negative integers with $\sum_{d|N}c_d\phi\left(\gcd\left(d,\frac{N}{d}\right)\right)=\frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]$. Let D

be the divisor

$$D = \sum_{d|N} c_d \left(\sum_j c_d \begin{bmatrix} j \\ d \end{bmatrix} \right) - \mathrm{div}(h(z)).$$

Here, the sum on j is over all $\Gamma_0(N)$-classes of cusps $\begin{bmatrix} j \\ d \end{bmatrix}$ with denominator d.

Then, D is a divisor supported at cusps and has degree zero. Since $X_0(N)$ has genus zero, $J_0(N)(\mathbb{Q})_{\text{cusp}}$ is trivial and every degree zero cuspidal divisor is the divisor of some modular function $i(z)$ with rational Fourier coefficients, and with leading Fourier coefficient 1. Let $f(z) = h(z)i(z)$. If C is the least common multiple of the denominators of the coefficients of $f(z)$, then $Cf(z) \in M_k(\Gamma_0(N)) \cap \mathbb{Z}[[q]]$ and is nonvanishing on $\mathbb{H}$. Corollary 18 implies that every coefficient is a multiple of C and therefore $C = 1$. Theorem 7 now implies that $f(z)$ is an eta-quotient, and we have that

$$\mathrm{div}(f) = \sum_{d|N} c_d \left(\sum_j \begin{bmatrix} j \\ d \end{bmatrix} \right).$$

□

References

1. V. Baldoni, N. Berline, J.A. De Loera, B. Dutra, M. Köppe, S. Moreinis, G. Pinto, M. Vergne, and J. Wu, *A user's guide for* `latte integrale` *v1.6, 2013, software package*, `LattE` is available at `http://www.math.ucdavis.edu/~latte/`.
2. Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993). MR 1484478
3. Richard Dedekind, *Schreiben an Herrn Borchardt über die Theorie der elliptischen Modul-Functionen*, J. Reine Angew. Math. **83** (1877), 265–292.
4. Fred Diamond and Jerry Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005. MR 2112196 (2006f:11045)
5. V. G. Drinfeld, *Two theorems on modular curves*, Funkcional. Anal. i Priložen. **7** (1973), no. 2, 83–84. MR 0318157 (47 #6705)
6. Kamal Khuri-Makdisi, *Linear algebra algorithms for divisors on an algebraic curve*, Math. Comp. **73** (2004), no. 245, 333–357 (electronic). MR 2034126 (2005a:14081)
7. Winfried Kohnen, *On a certain class of modular functions*, Proc. Amer. Math. Soc. **133** (2005), no. 1, 65–70 (electronic). MR 2085154 (2005d:11061)
8. Gérard Ligozat, *Courbes modulaires de genre* 1, Société Mathématique de France, Paris, 1975, Bull. Soc. Math. France, Mém. 43, Supplément au Bull. Soc. Math. France Tome 103, no. 3. MR 0417060 (54 #5121)
9. San Ling, *On the* **Q***-rational cuspidal subgroup and the component group of* $J_0(p^r)$, Israel J. Math. **99** (1997), 29–54. MR 1469086 (98e:11076)
10. Ju. I. Manin, *Parabolic points and zeta functions of modular curves*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 19–66. MR 0314846 (47 #3396)
11. David Mumford, *Varieties defined by quadratic equations*, Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969), Edizioni Cremonese, Rome, 1970, pp. 29–100. MR 0282975 (44 #209)

12. Morris Newman, *Construction and application of a class of modular functions*, Proc. London. Math. Soc. (3) **7** (1957), 334–350. MR 0091352 (19,953c)
13. ———, *Construction and application of a class of modular functions. II*, Proc. London Math. Soc. (3) **9** (1959), 373–387. MR 0107629 (21 #6354)
14. Ken Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics, vol. 102, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2004. MR 2020489 (2005c:11053)
15. Nadim Rustom, *Generators of graded rings of modular forms*, Preprint.
16. Goro Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, vol. 11, Princeton University Press, Princeton, NJ, 1994, Reprint of the 1971 original, Kanô Memorial Lectures, 1. MR 1291394 (95e:11048)
17. Glenn Stevens, *Arithmetic on modular curves*, Progress in Mathematics, vol. 20, Birkhäuser Boston Inc., Boston, MA, 1982. MR 670070 (87b:11050)

Department of Mathematics, Wake Forest University, Winston-Salem, NC 27109

E-mail address: rouseja@wfu.edu

Department of Mathematics, Wake Forest University, Winston-Salem, NC 27109, Department of Mathematics & Statistics, James Madison University, Harrisonburg, VA 22802

E-mail address: webbjj@jmu.edu