Integers represented by positive-definite quadratic forms - the modular approach

Jeremy Rouse

WAKE FOREST UNIVERSITY

Conference on aspects of the algebraic and analytic theory of quadratic forms
University of Georgia
July 23, 2017

Acknowledgements

- I'd like to thank the organizers of this conference for the opportunity to speak.

Acknowledgements

- I'd like to thank the organizers of this conference for the opportunity to speak.

- Some of the work I will mention was supported by NSF grant DMS-0901090.

Acknowledgements

- I'd like to thank the organizers of this conference for the opportunity to speak.

- Some of the work I will mention was supported by NSF grant DMS-0901090.

- I'd also like to thank the following people for very helpful conversations: Manjul Bhargava, Justin DeBenedetto, Noam Elkies, Jonathan Hanke, David Hansen, Will Jagy, Ben Kane, Ken Ono, and Katherine Thompson.

Outline

- Day 1

Outline

- Day 1
  - Universality theorems

Outline

- Day 1
  - Universality theorems
  - Escalation

Outline

- Day 1
    - Universality theorems
    - Escalation
    - Congruences conditions for representability

Outline

- Day 1
  - Universality theorems
  - Escalation
  - Congruences conditions for representability
  - Regular quadratic forms

Outline

- Day 1
  - Universality theorems
  - Escalation
  - Congruences conditions for representability
  - Regular quadratic forms
- Day 2

Outline

- Day 1
  - Universality theorems
  - Escalation
  - Congruences conditions for representability
  - Regular quadratic forms
- Day 2
  - Overview of modular forms

Outline

- Day 1
  - Universality theorems
  - Escalation
  - Congruences conditions for representability
  - Regular quadratic forms
- Day 2
  - Overview of modular forms
  - Theta series, Eisenstein series, newforms

Outline

- Day 1
  - Universality theorems
  - Escalation
  - Congruences conditions for representability
  - Regular quadratic forms
- Day 2
  - Overview of modular forms
  - Theta series, Eisenstein series, newforms
  - Determining the integers represented by a quadratic form

Outline

- Day 1
  - Universality theorems
  - Escalation
  - Congruences conditions for representability
  - Regular quadratic forms
- Day 2
  - Overview of modular forms
  - Theta series, Eisenstein series, newforms
  - Determining the integers represented by a quadratic form
- Day 3

Outline

- Day 1
  - Universality theorems
  - Escalation
  - Congruences conditions for representability
  - Regular quadratic forms
- Day 2
  - Overview of modular forms
  - Theta series, Eisenstein series, newforms
  - Determining the integers represented by a quadratic form
- Day 3
  - The Petersson inner product

Outline

- Day 1
  - Universality theorems
  - Escalation
  - Congruences conditions for representability
  - Regular quadratic forms
- Day 2
  - Overview of modular forms
  - Theta series, Eisenstein series, newforms
  - Determining the integers represented by a quadratic form
- Day 3
  - The Petersson inner product
  - L-functions

Outline

- Day 1
  - Universality theorems
  - Escalation
  - Congruences conditions for representability
  - Regular quadratic forms
- Day 2
  - Overview of modular forms
  - Theta series, Eisenstein series, newforms
  - Determining the integers represented by a quadratic form
- Day 3
  - The Petersson inner product
  - L-functions
  - Proof of a general theorem

History

- In 1770, Lagrange proved that every positive integer can be written as a sum of four squares.

History

- In 1770, Lagrange proved that every positive integer can be written as a sum of four squares.

- For example

$$234567 = 381^2 + 299^2 + 2^2 + 1^2.$$

History

- In 1770, Lagrange proved that every positive integer can be written as a sum of four squares.

- For example

$$234567 = 381^2 + 299^2 + 2^2 + 1^2.$$

- Q: What other expressions represent all positive integers?

Ramanujan

Ramanujan

- In 1916, Ramanujan claimed that there are precisely 55 4-tuples of positive integers (a, b, c, d) so that every positive integer is of the form

$$ax^2 + by^2 + cz^2 + dw^2.$$

Ramanujan

• In 1916, Ramanujan claimed that there are precisely 55 4-tuples of positive integers (a, b, c, d) so that every positive integer is of the form

$$ax^2 + by^2 + cz^2 + dw^2.$$

• In 1927, Dickson proved Ramanujan's claim (modulo one error). The form $x^2 + 2y^2 + 5z^2 + 5w^2$ represents every positive integer except 15.

Definitions (1/2)

- A quadratic form $Q(\vec{x})$ is called *integer-matrix* if

$$Q(\vec{x}) = \vec{x}^T A \vec{x}$$

where A is a matrix with integer entries.

Definitions (1/2)

- A quadratic form $Q(\vec{x})$ is called *integer-matrix* if

$$Q(\vec{x}) = \vec{x}^T A\vec{x}$$

where A is a matrix with integer entries.

- This means the cross terms must be even.

Definitions (1/2)

- A quadratic form $Q(\vec{x})$ is called *integer-matrix* if

$$Q(\vec{x}) = \vec{x}^T A \vec{x}$$

where A is a matrix with integer entries.

- This means the cross terms must be even.
- The form $x^2 + 2xy + 4y^2$ is an integer-matrix form.

Definitions (2/2)

- An *integer-valued* quadratic form Q can be written in the form

$$Q(\vec{x}) = \frac{1}{2}\vec{x}^T A\vec{x}$$

where A has integer entries and even diagonal entries.

Definitions (2/2)

• An *integer-valued* quadratic form Q can be written in the form

$$Q(\vec{x}) = \frac{1}{2}\vec{x}^T A\vec{x}$$

where A has integer entries and even diagonal entries.

• The form $x^2 + xy + 2y^2$ is an integer-valued form, but not an integer-matrix form.

Definitions (2/2)

- An *integer-valued* quadratic form Q can be written in the form

$$Q(\vec{x}) = \frac{1}{2}\vec{x}^T A\vec{x}$$

where A has integer entries and even diagonal entries.

- The form $x^2 + xy + 2y^2$ is an integer-valued form, but not an integer-matrix form.

- A quadratic form Q is positive-definite if $Q(\vec{x}) \geq 0$ for all $\vec{x} \in \mathbb{R}^n$, with equality if and only if $\vec{x} = \vec{0}$.

Classification

- In 1948, Margaret Willerding proved that there were exactly 178 integer-matrix quaternary quadratic forms that represented all positive integers, and gave a list of these.

Classification

- In 1948, Margaret Willerding proved that there were exactly 178 integer-matrix quaternary quadratic forms that represented all positive integers, and gave a list of these.

- In 2000, Bhargava determined that there are actually 204 integer-matrix quaternary forms that represent all positive integers.

Classification

- In 1948, Margaret Willerding proved that there were exactly 178 integer-matrix quaternary quadratic forms that represented all positive integers, and gave a list of these.

- In 2000, Bhargava determined that there are actually 204 integer-matrix quaternary forms that represent all positive integers.

- Apparently, Willerding had missed 36 forms, listed one twice, and listed 9 forms that fail to represent all positive integers.

Universality theorems

Theorem (The 15-theorem, Conway-Schneeberger)

A positive-definite, integer-matrix form Q represents every positive integer if and only if it represents 1, 2, 3, 5, 6, 7, 10, 14, *and* 15.

Universality theorems

Theorem (The 15-theorem, Conway-Schneeberger)

A positive-definite, integer-matrix form Q represents every positive integer if and only if it represents 1, 2, 3, 5, 6, 7, 10, 14, *and* 15.

Theorem (The 290-theorem, Bhargava-Hanke)

A positive-definite, integer-valued form Q represents every positive integer if and only if it represents

$$1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29,$$
$$30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, \textit{and } 290.$$

Consequences

- Each of these results is sharp. The form

$$x^2 + 2y^2 + 4z^2 + 29w^2 + 145v^2 - xz - yz$$

represents every positive integer except 290.

Consequences

- Each of these results is sharp. The form

$$x^2 + 2y^2 + 4z^2 + 29w^2 + 145v^2 - xz - yz$$

represents every positive integer except 290.

- If a form represents every positive integer less than 290, it represents every integer greater than 290.

Consequences

- Each of these results is sharp. The form

$$x^2 + 2y^2 + 4z^2 + 29w^2 + 145v^2 - xz - yz$$

represents every positive integer except 290.

- If a form represents every positive integer less than 290, it represents every integer greater than 290.

- There are 6436 integer-valued quaternary forms that represent all positive integers.

More generality

Theorem (Bhargava)

Given an infinite set S of positive integers, there is a unique minimal finite subset $S_0 \subseteq S$ with the property that

$$Q \text{ represents everything in } S \iff Q \text{ represents everything in } S_0.$$

More generality

Theorem (Bhargava)

Given an infinite set S of positive integers, there is a unique minimal finite subset $S_0 \subseteq S$ with the property that

$$Q \text{ represents everything in } S \iff Q \text{ represents everything in } S_0.$$

- We say that a quadratic form Q is *S-universal* if Q represents everything in S.

More generality

Theorem (Bhargava)

Given an infinite set S of positive integers, there is a unique minimal finite subset $S_0 \subseteq S$ with the property that

$$Q \text{ represents everything in } S \iff Q \text{ represents everything in } S_0.$$

- We say that a quadratic form Q is S*-universal* if Q represents everything in S.
- Given a set S, how does one find the set S_0?

Later results

Theorem (The 451-theorem, R, 2014)

Assume GRH. Then a positive-definite, integer-valued form Q represents all positive odds if and only if it represents

$$1, 3, 5, 7, 11, 13, 15, 17, 19, 21, 23, 29, 31, 33, 35, 37, 39, 41, 47, 51, 53, 57, 59, 77, 83, 85, 87, 89, 91, 93, 105, 119, 123, 133, 137, 143, 145, 187, 195, 203, 205, 209, 231, 319, 385, \textit{and } 451.$$

Later results

Theorem (The 451-theorem, R, 2014)

Assume GRH. Then a positive-definite, integer-valued form Q represents all positive odds if and only if it represents

$$1, 3, 5, 7, 11, 13, 15, 17, 19, 21, 23, 29, 31, 33, 35, 37, 39, 41, 47,$$
$$51, 53, 57, 59, 77, 83, 85, 87, 89, 91, 93, 105, 119, 123, 133, 137,$$
$$143, 145, 187, 195, 203, 205, 209, 231, 319, 385, \textit{and } 451.$$

Theorem (DeBenedetto-R, to appear in *Ram. Journal*)

A positive-definite, integer-valued form Q represents every positive integer coprime to 3 if and only if it represents

$$1, 2, 5, 7, 10, 11, 13, 14, 17, 19, 22, 23, 26, 29, 31, 34, 35$$
$$37, 38, 46, 47, 55, 58, 62, 70, 94, 110, 119, 145, 203, \textit{and } 290.$$

Two exceptions

- It follows from the proof of the 15-theorem that if an integer-valued form Q represents all positive integers with one exception, then that exception must be 1, 2, 3, 5, 6, 7, 10, 14, or 15.

Two exceptions

- It follows from the proof of the 15-theorem that if an integer-valued form Q represents all positive integers with one exception, then that exception must be 1, 2, 3, 5, 6, 7, 10, 14, or 15.

Theorem (BDMSST, to appear in *Proc. Amer. Math. Soc.*)

If a positive-definite integer-matrix form Q represents all positive integers with two exceptions, the pair of exceptions $\{m, n\}$ must be one of the following: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{1,6\}$, $\{1,7\}$, $\{1,9\}$, $\{1,10\}$, $\{1,11\}$, $\{1,13\}$, $\{1,14\}$, $\{1,15\}$, $\{1,17\}$, $\{1,19\}$, $\{1,21\}$, $\{1,23\}$, $\{1,25\}$, $\{1,30\}$, $\{1,41\}$, $\{1,55\}$, $\{2,3\}$, $\{2,5\}$, $\{2,6\}$, $\{2,8\}$, $\{2,10\}$, $\{2,11\}$, $\{2,14\}$, $\{2,15\}$, $\{2,18\}$, $\{2,22\}$, $\{2,30\}$, $\{2,38\}$, $\{2,50\}$, $\{3,6\}$, $\{3,7\}$, $\{3,11\}$, $\{3,12\}$, $\{3,19\}$, $\{3,21\}$, $\{3,27\}$, $\{3,30\}$, $\{3,35\}$, $\{3,39\}$, $\{5,7\}$, $\{5,10\}$, $\{5,13\}$, $\{5,14\}$, $\{5,20\}$, $\{5,21\}$, $\{5,29\}$, $\{5,30\}$, $\{5,35\}$, $\{5,37\}$, $\{5,42\}$, $\{5,125\}$, $\{6,15\}$, $\{6,54\}$, $\{7,10\}$, $\{7,15\}$, $\{7,23\}$, $\{7,28\}$, $\{7,31\}$, $\{7,39\}$, $\{7,55\}$, $\{10,15\}$, $\{10,26\}$, $\{10,40\}$, $\{10,58\}$, $\{10,250\}$, $\{14,30\}$, $\{14,56\}$, $\{14,78\}$.

Lattices

- A *lattice* L of dimension n is a discrete subgroup of $\mathbb{R}^n$ that is isomorphic to $\mathbb{Z}^n$.

Lattices

- A *lattice* L of dimension n is a discrete subgroup of $\mathbb{R}^n$ that is isomorphic to $\mathbb{Z}^n$.

- A lattice comes with a positive definite inner product $\langle \cdot, \cdot \rangle$.

Lattices

- A *lattice* L of dimension n is a discrete subgroup of $\mathbb{R}^n$ that is isomorphic to $\mathbb{Z}^n$.

- A lattice comes with a positive definite inner product $\langle \cdot, \cdot \rangle$.

- Given a lattice L, the function $Q(\vec{x}) = \langle \vec{x}, \vec{x} \rangle$ is a quadratic form.

Lattices

- A *lattice* L of dimension n is a discrete subgroup of $\mathbb{R}^n$ that is isomorphic to $\mathbb{Z}^n$.

- A lattice comes with a positive definite inner product $\langle \cdot, \cdot \rangle$.

- Given a lattice L, the function $Q(\vec{x}) = \langle \vec{x}, \vec{x} \rangle$ is a quadratic form.

- Conversely, given a quadratic form Q, one can associate to it a lattice $L \cong \mathbb{Z}^4$ by defining

$$\langle \vec{x}, \vec{y} \rangle = \frac{1}{2} \left(Q(\vec{x} + \vec{y}) - Q(\vec{x}) - Q(\vec{y}) \right).$$

Escalation

• If Q is a quadratic form (with corresponding lattice L) is not S-universal, we call the *truant* of Q/L, the smallest element $t \in S$ that is not represented by Q.

Escalation

- If Q is a quadratic form (with corresponding lattice L) is not S-universal, we call the *truant* of Q/L, the smallest element $t \in S$ that is not represented by Q.

- An *escalation* of L is a lattice L' generated by L and a vector of norm t.

Escalation

- If Q is a quadratic form (with corresponding lattice L) is not S-universal, we call the *truant* of Q/L, the smallest element $t \in S$ that is not represented by Q.

- An *escalation* of L is a lattice L' generated by L and a vector of norm t.

- An *escalator lattice* is a lattice obtained by repeatedly escalating the zero-dimensional lattice.

Example (1/3)

- Let $S = \{1, 3, 5, 7, \ldots\}$ be the set of odd numbers. There is a unique one-dimensional escalator lattice corresponding to x^2.

Example (1/3)

- Let $S = \{1, 3, 5, 7, \ldots\}$ be the set of odd numbers. There is a unique one-dimensional escalator lattice corresponding to x^2.

- This quadratic form has truant 3, and its escalations are $x^2 + axy + 3y^2$. To be positive-definite we must have $-3 \leq a \leq 3$.

Example (1/3)

- Let $S = \{1, 3, 5, 7, \ldots\}$ be the set of odd numbers. There is a unique one-dimensional escalator lattice corresponding to x^2.

- This quadratic form has truant 3, and its escalations are $x^2 + axy + 3y^2$. To be positive-definite we must have $-3 \leq a \leq 3$.

- Each of $x^2 + 3y^2$, $x^2 + xy + 3y^2$, $x^2 + 2xy + 3y^2$ and $x^2 + 3xy + 3y^2$ has truant 5 or 7.

Example (2/3)

- Escalating the four forms listed previously gives 73 three-dimensional lattices.

Example (2/3)

- Escalating the four forms listed previously gives 73 three-dimensional lattices.

- Of these, 50 of these fail to represent some odd number ≤ 73. The remaining 23 represent all odd numbers $\leq 10^6$.

Example (2/3)

- Escalating the four forms listed previously gives 73 three-dimensional lattices.

- Of these, 50 of these fail to represent some odd number ≤ 73. The remaining 23 represent all odd numbers $\leq 10^6$.

- Escalating the 50 lattices that are definitely not S-universal yields 24312 four-dimensional escalator lattices.

Example (3/3)

- It can be tricky to determine whether a lattice is S-universal or not.

Example (3/3)

- It can be tricky to determine whether a lattice is S-universal or not.

- In particular, it is not known whether or not

$$x^2 + 2y^2 + 5z^2 + xy$$
$$x^2 + 3y^2 + 6z^2 + xy + 2yz$$
$$x^2 + 3y^2 + 7z^2 + xy + xz$$

each represents all odd numbers.

Example (3/3)

• It can be tricky to determine whether a lattice is S-universal or not.

• In particular, it is not known whether or not

$$x^2 + 2y^2 + 5z^2 + xy$$
$$x^2 + 3y^2 + 6z^2 + xy + 2yz$$
$$x^2 + 3y^2 + 7z^2 + xy + xz$$

each represents all odd numbers.

• One can show that GRH implies that each of the above three forms represents all odd numbers.

S-universal escalators

- Fact 1: A lattice L is S-universal if and only if it has a sublattice L' which is an S-universal escalator lattice.

S-universal escalators

- Fact 1: A lattice L is S-universal if and only if it has a sublattice L' which is an S-universal escalator lattice.

- Proof: Suppose that L is S-universal. If $S = \{t_1, t_2, t_3, \ldots\}$, let L_1 be a sublattice of L generated by a vector of norm t_1. For $i \geq 2$, let L_i be a lattice containing L_{i-1} that represents t_i.

S-universal escalators

- Fact 1: A lattice L is S-universal if and only if it has a sublattice L' which is an S-universal escalator lattice.

- Proof: Suppose that L is S-universal. If $S = \{t_1, t_2, t_3, \dots\}$, let L_1 be a sublattice of L generated by a vector of norm t_1. For $i \geq 2$, let L_i be a lattice containing L_{i-1} that represents t_i.

- The ascending chain $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots \subseteq L$ must stabilize. It stabilizes in an S-universal escalator lattice.

More about escalators

- Fact 2: The set S_0 in Bhargava's theorem is precisely the set of truants of escalator lattices.

More about escalators

- Fact 2: The set S_0 in Bhargava's theorem is precisely the set of truants of escalator lattices.

- The hard part is, given a quadratic form Q, determining whether it is S-universal or not.

More about escalators

- Fact 2: The set S_0 in Bhargava's theorem is precisely the set of truants of escalator lattices.

- The hard part is, given a quadratic form Q, determining whether it is S-universal or not.

- Exercise 1: Suppose that Q is a positive-definite quadratic form. Assume that Q represents 2, and Q also represents 3. Show that Q also represents 818.

More about escalators

- Fact 2: The set S_0 in Bhargava's theorem is precisely the set of truants of escalator lattices.

- The hard part is, given a quadratic form Q, determining whether it is S-universal or not.

- Exercise 1: Suppose that Q is a positive-definite quadratic form. Assume that Q represents 2, and Q also represents 3. Show that Q also represents 818.

- Exercise 2: Let $S = \mathbb{N}$ be the set of positive integers. Show that there is no positive-definite S-universal ternary quadratic form.

Necessary conditions

- In order for there to be a solution to $Q(x_1, x_2, \ldots, x_r) = n$ with $x_i \in \mathbb{Z}$,

Necessary conditions

- In order for there to be a solution to $Q(x_1, x_2, \ldots, x_r) = n$ with $x_i \in \mathbb{Z}$, it must be that for all $m \geq 1$, there is a solution to $Q(x_1, x_2, \ldots, x_r) = n$ with $x_i \in \mathbb{Z}/m\mathbb{Z}$.

Necessary conditions

- In order for there to be a solution to $Q(x_1, x_2, \ldots, x_r) = n$ with $x_i \in \mathbb{Z}$, it must be that for all $m \geq 1$, there is a solution to $Q(x_1, x_2, \ldots, x_r) = n$ with $x_i \in \mathbb{Z}/m\mathbb{Z}$.

- For example, $Q(x, y, z, w) = x^2 + y^2 + z^2 + 8w^2$ does not represent any $n \equiv 7 \pmod{8}$ because there are no solutions to

$$7 \equiv x^2 + y^2 + z^2 + 8w^2 \pmod{8}.$$

Necessary conditions

- In order for there to be a solution to $Q(x_1, x_2, \ldots, x_r) = n$ with $x_i \in \mathbb{Z}$, it must be that for all $m \geq 1$, there is a solution to $Q(x_1, x_2, \ldots, x_r) = n$ with $x_i \in \mathbb{Z}/m\mathbb{Z}$.

- For example, $Q(x, y, z, w) = x^2 + y^2 + z^2 + 8w^2$ does not represent any $n \equiv 7 \pmod 8$ because there are no solutions to

$$7 \equiv x^2 + y^2 + z^2 + 8w^2 \pmod 8.$$

- It turns out that Q represents every positive integer that is not congruent to $7 \pmod 8$.

p-adic numbers

- For $x \in \mathbb{Q}$ and a prime number p, write $x = p^k \cdot \frac{a}{b}$ where $\gcd(a, b) = 1$ and $p \nmid a$ and $p \nmid b$.

p-adic numbers

- For $x \in \mathbb{Q}$ and a prime number p, write $x = p^k \cdot \frac{a}{b}$ where $\gcd(a, b) = 1$ and $p \nmid a$ and $p \nmid b$.
- Define $|x|_p = p^{-k}$. Define a metric on $\mathbb{Q}$ by $d(x, y) = |x - y|_p$.

p-adic numbers

- For $x \in \mathbb{Q}$ and a prime number p, write $x = p^k \cdot \frac{a}{b}$ where $\gcd(a, b) = 1$ and $p \nmid a$ and $p \nmid b$.

- Define $|x|_p = p^{-k}$. Define a metric on $\mathbb{Q}$ by $d(x, y) = |x - y|_p$.

- Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to this metric and $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$.

Local stuff

- We say that a quadratic form Q *locally represents* $n > 0$ if, for all primes p, there is a solution to $Q(\vec{x}) = n$ with $\vec{x} \in \mathbb{Z}_p^r$.

Local stuff

- We say that a quadratic form Q *locally represents* $n > 0$ if, for all primes p, there is a solution to $Q(\vec{x}) = n$ with $\vec{x} \in \mathbb{Z}_p^r$.

- We say that Q_1 and Q_2 are *locally equivalent* if $Q_i = \frac{1}{2}\vec{x}^T A_i \vec{x}$ and for all primes p, there is a matrix $M \in \mathrm{GL}_r(\mathbb{Z}_p)$ so that

$$MA_1M^T = A_2.$$

Local stuff

- We say that a quadratic form Q *locally represents* $n > 0$ if, for all primes p, there is a solution to $Q(\vec{x}) = n$ with $\vec{x} \in \mathbb{Z}_p^r$.

- We say that Q_1 and Q_2 are *locally equivalent* if $Q_i = \frac{1}{2}\vec{x}^T A_i \vec{x}$ and for all primes p, there is a matrix $M \in \mathrm{GL}_r(\mathbb{Z}_p)$ so that

$$MA_1M^T = A_2.$$

Theorem (Hasse-Minkowski)

Suppose that Q is a positive-definite quadratic form and n is locally represented by Q. Then there is some $\vec{x} \in \mathbb{Q}^r$ so that $Q(\vec{x}) = n$.

The genus

- If Q is a positive-definite quadratic form, we let $\mathrm{Gen}(Q)$ denote the set of QFs that are locally equivalent to Q.

The genus

- If Q is a positive-definite quadratic form, we let $\mathrm{Gen}(Q)$ denote the set of QFs that are locally equivalent to Q.

Theorem

The set $\mathrm{Gen}(Q)$ *is finite.*

The genus

- If Q is a positive-definite quadratic form, we let $\mathrm{Gen}(Q)$ denote the set of QFs that are locally equivalent to Q.

Theorem

The set $\mathrm{Gen}(Q)$ *is finite.*

Theorem

If n is locally represented by Q, then there is at least one form $R \in \mathrm{Gen}(Q)$ so that R represents Q.

Example

- Let $Q_1 = x^2 + 3y^2 + 3z^2 + xy + 3yz$. Then $\mathrm{Gen}(Q_1)$ consists of two forms.

Example

- Let $Q_1 = x^2 + 3y^2 + 3z^2 + xy + 3yz$. Then $\mathrm{Gen}(Q_1)$ consists of two forms.

- The other form is $Q_2 = x^2 + xy + y^2 + 8z^2$.

Example

- Let $Q_1 = x^2 + 3y^2 + 3z^2 + xy + 3yz$. Then $\mathrm{Gen}(Q_1)$ consists of two forms.

- The other form is $Q_2 = x^2 + xy + y^2 + 8z^2$.

- Note: If there is a genus $\mathrm{Gen}(Q)$ consisting of a single form, that form is guaranteed to represent all n that are locally represented by Q.

Tartakowski's theorem

- The Hasse-Minkowski theorem does not guarantee that n is actually represented by Q.

Tartakowski's theorem

- The Hasse-Minkowski theorem does not guarantee that n is actually represented by Q.

Theorem (Tartakowski)

Suppose that Q is a positive-definite quadratic form in $r \geq 5$ variables. Then every sufficiently large locally represented positive integer is represented by Q.

What happens for $r = 4$?

- Let $Q(x, y, z, w) = x^2 + y^2 + 7z^2 + 7w^2$. Then Q locally represents every positive integer.

What happens for $r = 4$?

- Let $Q(x, y, z, w) = x^2 + y^2 + 7z^2 + 7w^2$. Then Q locally represents every positive integer.

- However, if $Q(x, y, z, w) \equiv 0 \pmod{49}$, then $x \equiv y \equiv z \equiv w \pmod{7}$.

What happens for $r = 4$?

- Let $Q(x, y, z, w) = x^2 + y^2 + 7z^2 + 7w^2$. Then Q locally represents every positive integer.

- However, if $Q(x, y, z, w) \equiv 0 \pmod{49}$, then $x \equiv y \equiv z \equiv w \pmod{7}$.

- It follows that Q does not represent $3 \cdot 49^n$ for any $n \geq 0$.

Anisotropic primes

- We say that a quadratic form Q is anisotropic at the prime p if whenever $\vec{x} \in \mathbb{Z}_p^r$ and $Q(\vec{x}) = 0$, then $\vec{x} = \vec{0}$.

Anisotropic primes

- We say that a quadratic form Q is anisotropic at the prime p if whenever $\vec{x} \in \mathbb{Z}_p^r$ and $Q(\vec{x}) = 0$, then $\vec{x} = \vec{0}$.

- If Q is anisotropic at p, then $r \leq 4$.

Anisotropic primes

- We say that a quadratic form Q is anisotropic at the prime p if whenever $\vec{x} \in \mathbb{Z}_p^r$ and $Q(\vec{x}) = 0$, then $\vec{x} = \vec{0}$.

- If Q is anisotropic at p, then $r \leq 4$.

Theorem

Suppose that Q is a four-variable quadratic form. Then there is a constant $C(Q)$ so that if $n > C(Q)$ is locally represented by Q, then either n is represented by Q, or there is an anisotropic prime p so that $p^2 | n$ and n/p^2 is not represented by Q.

A three-variable phenomenon

- Let $Q(x, y, z) = 3x^2 + 4y^2 + 9z^2$.

A three-variable phenomenon

- Let $Q(x, y, z) = 3x^2 + 4y^2 + 9z^2$.

- This form locally represents n provided $n \not\equiv 2 \pmod 4$ and $n \not\equiv 2 \cdot 3^{\alpha-1} \pmod{3^\alpha}$ for any α.

A three-variable phenomenon

- Let $Q(x, y, z) = 3x^2 + 4y^2 + 9z^2$.
- This form locally represents n provided $n \not\equiv 2 \pmod 4$ and $n \not\equiv 2 \cdot 3^{\alpha-1} \pmod{3^\alpha}$ for any α.
- Any perfect square is locally represented by Q.

A three-variable phenomenon

- Let $Q(x, y, z) = 3x^2 + 4y^2 + 9z^2$.

- This form locally represents n provided $n \not\equiv 2 \pmod 4$ and $n \not\equiv 2 \cdot 3^{\alpha-1} \pmod{3^\alpha}$ for any α.

- Any perfect square is locally represented by Q.

- However Q does not represent n^2 if all prime factors of n are $\equiv 1 \pmod 3$.

How many local solutions?

- We can gather more precise local information to try to estimate the number of times an integer n "should" be represented by Q.

How many local solutions?

- We can gather more precise local information to try to estimate the number of times an integer n "should" be represented by Q.

- If Q is a form, n is a positive integer, and p is a prime, define

$$\beta_p(n) = \lim_{U \to \{n\}} \frac{\mathrm{Vol}(Q^{-1}(U))}{\mathrm{Vol}(U)}.$$

How many local solutions?

- We can gather more precise local information to try to estimate the number of times an integer n "should" be represented by Q.

- If Q is a form, n is a positive integer, and p is a prime, define

$$\beta_p(n) = \lim_{U \to \{n\}} \frac{\mathrm{Vol}(Q^{-1}(U))}{\mathrm{Vol}(U)}.$$

- More concretely, this is

$$\beta_p(n) = \lim_{v \to \infty} \frac{\#\{\vec{x} \in (\mathbb{Z}/p^v\mathbb{Z})^r : Q(\vec{x}) \equiv n \pmod{p^v}\}}{p^{(r-1)v}}.$$

Local densities

- If $p = \infty$, and $Q = \frac{1}{2}\vec{x}^T A\vec{x}$, then

$$\beta_{\infty}(n) = \frac{4\pi^2 n}{\sqrt{\det(A)}}.$$

Local densities

- If $p = \infty$, and $Q = \frac{1}{2}\vec{x}^T A \vec{x}$, then

$$\beta_\infty(n) = \frac{4\pi^2 n}{\sqrt{\det(A)}}.$$

- Computing $\beta_p(n)$ can be tricky in general. There are explicit formulas for the $\beta_p(n)$ given in Yang's 1998 paper in the Journal of Number Theory.

Local densities

- If $p = \infty$, and $Q = \frac{1}{2}\vec{x}^T A\vec{x}$, then

$$\beta_\infty(n) = \frac{4\pi^2 n}{\sqrt{\det(A)}}.$$

- Computing $\beta_p(n)$ can be tricky in general. There are explicit formulas for the $\beta_p(n)$ given in Yang's 1998 paper in the Journal of Number Theory.

- The earliest work on quadratic forms was done via the circle method, and

$$\prod_{p \leq \infty} \beta_p(n)$$

is the "main term" approximation for $r_Q(n)$.

Definition

- A quadratic form Q is called *regular* if every locally represented integer is represented.

Definition

- A quadratic form Q is called *regular* if every locally represented integer is represented.

- As noted above, if $\mathrm{Gen}(Q)$ consists of only one class, then Q is automatically regular.

Definition

- A quadratic form Q is called *regular* if every locally represented integer is represented.

- As noted above, if $\mathrm{Gen}(Q)$ consists of only one class, then Q is automatically regular.

Theorem (Kaplansky, 1995)

The form $Q = x^2 + 3y^2 + 3z^2 + xy + 3yz$ *is regular.*

Proof of Kaplansky's theorem (1/2)

Lemma

If $n = x^2 + xy + y^2$, then there are integers r and s so that $n = r^2 + 3s^2$.

Proof of Kaplansky's theorem (1/2)

Lemma

If $n = x^2 + xy + y^2$, then there are integers r and s so that $n = r^2 + 3s^2$.

- If one of x or y is even (say x with $x = 2k$), we have $n = 4k^2 + 2ky + y^2 = (k + y)^2 + 3k^2$.

Proof of Kaplansky's theorem (1/2)

Lemma

If $n = x^2 + xy + y^2$, then there are integers r and s so that $n = r^2 + 3s^2$.

- If one of x or y is even (say x with $x = 2k$), we have $n = 4k^2 + 2ky + y^2 = (k + y)^2 + 3k^2$.

- If x and y are both odd, we rewrite $n = x^2+xy+y^2 = (x+y)^2+(x+y)(-x)+(-x)^2 = A^2+AB+B^2$.

Proof of Kaplansky's theorem (2/2)

- Assume that n is locally represented by $Q = x^2 + 3y^2 + 3z^2 + xy + 3yz$. Then either n is represented by Q, or by $R = x^2 + xy + y^2 + 8z^2$, the other form in $\mathrm{Gen}(Q)$.

Proof of Kaplansky's theorem (2/2)

- Assume that n is locally represented by $Q = x^2 + 3y^2 + 3z^2 + xy + 3yz$. Then either n is represented by Q, or by $R = x^2 + xy + y^2 + 8z^2$, the other form in $\mathrm{Gen}(Q)$.

- Assume that $R = x^2 + xy + y^2 + 8z^2$ represents n. Then, there are $r, s \in \mathbb{Z}$ so that $n = r^2 + 3s^2 + 8z^2$.

Proof of Kaplansky's theorem (2/2)

- Assume that n is locally represented by $Q = x^2 + 3y^2 + 3z^2 + xy + 3yz$. Then either n is represented by Q, or by $R = x^2 + xy + y^2 + 8z^2$, the other form in $\mathrm{Gen}(Q)$.

- Assume that $R = x^2 + xy + y^2 + 8z^2$ represents n. Then, there are $r, s \in \mathbb{Z}$ so that $n = r^2 + 3s^2 + 8z^2$.

- A simple calculation shows that $Q(r - z, 2z, s - z) = n$. This proves that Q is regular.

Regular ternary quadratic forms

- In 1997, Jagy, Kaplansky, and Schiemann proved that there are at most 913 regular ternary quadratic forms.

Regular ternary quadratic forms

- In 1997, Jagy, Kaplansky, and Schiemann proved that there are at most 913 regular ternary quadratic forms.

- Of their 913 candidates, they proved that 891 of them were regular.

Regular ternary quadratic forms

- In 1997, Jagy, Kaplansky, and Schiemann proved that there are at most 913 regular ternary quadratic forms.

- Of their 913 candidates, they proved that 891 of them were regular.

- In 2011, Oh proved that 8 more of their candidates were regular. In 2014, Lemke-Oliver proved the remaining 14 were regular assuming GRH.

Using regular ternary forms

- Sometimes, a regular ternary form T can be used to determine which integers in S are represented by a quaternary form Q.

Using regular ternary forms

- Sometimes, a regular ternary form T can be used to determine which integers in S are represented by a quaternary form Q.

- Let L be the quaternary lattice. Suppose that L has a sublattice L' so that
  - L' corresponds to a regular ternary quadratic form,

Using regular ternary forms

- Sometimes, a regular ternary form T can be used to determine which integers in S are represented by a quaternary form Q.

- Let L be the quaternary lattice. Suppose that L has a sublattice L' so that
  - L' corresponds to a regular ternary quadratic form,
  - $L' \oplus (L')^{\perp}$ locally represents everything in S.

Using regular ternary forms

• Sometimes, a regular ternary form T can be used to determine which integers in S are represented by a quaternary form Q.

• Let L be the quaternary lattice. Suppose that L has a sublattice L' so that

- L' corresponds to a regular ternary quadratic form,
- $L' \oplus (L')^{\perp}$ locally represents everything in S.

• Then a simple calculation will determine the integers in S that are represented by L.

Example I

- Let $Q(x, y, z, w) = x^2 + y^2 + yz + 2z^2 + 7w^2$. The form $T(x, y, z) = x^2 + y^2 + yz + 2z^2$ is regular.

Example I

- Let $Q(x, y, z, w) = x^2 + y^2 + yz + 2z^2 + 7w^2$. The form $T(x, y, z) = x^2 + y^2 + yz + 2z^2$ is regular.

- The form T represents all positive integers except those $\equiv 21, 35, 42 \pmod{49}$.

Example I

- Let $Q(x, y, z, w) = x^2 + y^2 + yz + 2z^2 + 7w^2$. The form $T(x, y, z) = x^2 + y^2 + yz + 2z^2$ is regular.

- The form T represents all positive integers except those $\equiv 21, 35, 42 \pmod{49}$.

- Since

$$21 \equiv 7 \cdot 1^2 + 14 \pmod{49}$$
$$35 \equiv 7 \cdot 2^2 + 7 \pmod{49}$$
$$42 \equiv 7 \cdot 2^2 + 14 \pmod{49},$$

Q represents all positive integers.

Example II

- Let $Q(x,y,z,w) = x^2 + xy + 3y^2 + 4z^2 + 77w^2$. The form $T(x,y,z) = x^2 + xy + 3y^2 + 4z^2$ is regular.

Example II

- Let $Q(x,y,z,w) = x^2 + xy + 3y^2 + 4z^2 + 77w^2$. The form $T(x,y,z) = x^2 + xy + 3y^2 + 4z^2$ is regular.

- The form T is regular, and fails to represent only those n with $n \equiv 2 \pmod{4}$ and $n = 11^{\alpha}\beta$ with α odd and $\left(\frac{\beta}{11}\right) = -1$.

Example II

- Let $Q(x,y,z,w) = x^2 + xy + 3y^2 + 4z^2 + 77w^2$. The form $T(x,y,z) = x^2 + xy + 3y^2 + 4z^2$ is regular.

- The form T is regular, and fails to represent only those n with $n \equiv 2 \pmod 4$ and $n = 11^{\alpha}\beta$ with α odd and $\left(\frac{\beta}{11}\right) = -1$.

- A computer program needs to check 235 residue classes. It finds that Q represents all odd numbers except

$$143, 187, 231, 385, 451, 627, 935, 1111, 1419, 1903, \text{ and } 2387.$$

Proving the 451-theorem

- This method of using regular forms is a key method to proving the 451-theorem.

Proving the 451-theorem

- This method of using regular forms is a key method to proving the 451-theorem.

- There are 24312 four-dimensional escalators, and one must understand the odd integers represented by each.

Proving the 451-theorem

- This method of using regular forms is a key method to proving the 451-theorem.

- There are 24312 four-dimensional escalators, and one must understand the odd integers represented by each.

- This method of regular ternary forms can be used to handle about 7000 of the 24312.

An exercise

- The form $T(x,y,z) = x^2 + y^2 + z^2$ is regular. It represents all positive integers not of the form $4^k(8\ell + 7)$.

An exercise

- The form $T(x, y, z) = x^2 + y^2 + z^2$ is regular. It represents all positive integers not of the form $4^k(8\ell + 7)$.

- Exercise 3: Let p be a prime number and $Q(x, y, z, w) = x^2 + py^2 + pz^2 + pw^2$.

An exercise

- The form $T(x, y, z) = x^2 + y^2 + z^2$ is regular. It represents all positive integers not of the form $4^k(8\ell + 7)$.

- Exercise 3: Let p be a prime number and $Q(x, y, z, w) = x^2 + py^2 + pz^2 + pw^2$.

- Show that if $p \not\equiv 1 \pmod 8$, then every positive integer n which is congruent to a square mod p and $n > p(4p - 5)$ is represented by Q.

An exercise

- The form $T(x, y, z) = x^2 + y^2 + z^2$ is regular. It represents all positive integers not of the form $4^k(8\ell + 7)$.

- Exercise 3: Let p be a prime number and $Q(x, y, z, w) = x^2 + py^2 + pz^2 + pw^2$.

- Show that if $p \not\equiv 1 \pmod 8$, then every positive integer n which is congruent to a square mod p and $n > p(4p - 5)$ is represented by Q.

- Show that if $p \equiv 3 \pmod 8$, then $n = p(4p - 5)$ is not represented by Q.