

Notes for Lecture #8

Finite element method

The finite element approach is based on an expansion of the unknown electrostatic potential in terms of known functions of fixed shape $\{\phi_{ij}(x, y)\}$. In two dimensions this expansion takes the form:

$$4\pi\epsilon_0\Phi(x, y) = \sum_{ij} \psi_{ij}\phi_{ij}(x, y), \quad (1)$$

where ψ_{ij} represents the amplitude associated with the shape function $\phi_{ij}(x, y)$. The amplitude values can be determined for a given solution of the Poisson equation:

$$-\nabla^2 (4\pi\epsilon_0\Phi(x, y)) = 4\pi\rho(x, y), \quad (2)$$

by solving a linear algebra problem of the form

$$\sum_{ij} M_{kl,ij}\psi_{ij} = G_{kl}, \quad (3)$$

where

$$M_{kl,ij} \equiv \int dx \int dy \nabla\phi_{kl}(x, y) \cdot \nabla\phi_{ij}(x, y) \quad \text{and} \quad G_{kl} \equiv \int dx \int dy \phi_{kl}(x, y) 4\pi\rho(x, y). \quad (4)$$

In obtaining this result, we have assumed that the boundary values vanish. In order for this result to be useful, we need to be able evaluate the integrals for $M_{kl,ij}$ and for G_{kl} . In the latter case, we need to know the form of the charge density. The form of $M_{kl,ij}$ only depends upon the form of the shape functions. If we take these functions to be:

$$\phi_{ij}(x, y) \equiv \mathcal{X}_i(x)\mathcal{Y}_j(y), \quad (5)$$

where

$$\mathcal{X}_i(x) \equiv \begin{cases} \left(1 - \frac{|x-x_i|}{h}\right) & \text{for } x_i - h \leq x \leq x_i + h \\ 0 & \text{otherwise} \end{cases}, \quad (6)$$

and $\mathcal{Y}_j(y)$ has a similar expression in the variable y . Then

$$M_{kl,ij} \equiv \int dx \int dy \left[\frac{d\mathcal{X}_k(x)}{dx} \frac{d\mathcal{X}_i(x)}{dx} \mathcal{Y}_l(y)\mathcal{Y}_j(y) + \mathcal{X}_k(x)\mathcal{X}_i(x) \frac{d\mathcal{Y}_l(y)}{dy} \frac{d\mathcal{Y}_j(y)}{dy} \right]. \quad (7)$$

There are four types of non-trivial contributions to these values:

$$\int_{x_i-h}^{x_i+h} (\mathcal{X}_i(x))^2 dx = h \int_{-1}^1 (1 - |u|)^2 du = \frac{2h}{3}, \quad (8)$$

$$\int_{x_i-h}^{x_i+h} (\mathcal{X}_i(x)\mathcal{X}_{i+1}(x)) dx = h \int_0^1 (1 - u)u du = \frac{h}{6}, \quad (9)$$

$$\int_{x_i-h}^{x_i+h} \left(\frac{d\mathcal{X}_i(x)}{dx} \right)^2 dx = \frac{1}{h} \int_{-1}^1 du = \frac{2}{h}, \quad (10)$$

and

$$\int_{x_i-h}^{x_i+h} \left(\frac{d\mathcal{X}_i(x)}{dx} \frac{d\mathcal{X}_{i+1}(x)}{dx} \right) dx = -\frac{1}{h} \int_0^1 du = \frac{-1}{h}. \quad (11)$$

These basic ingredients lead to the following distinct values for the matrix:

$$M_{kl,ij} = \begin{cases} \frac{8}{3} & \text{for } k = i \text{ and } l = j \\ -\frac{1}{3} & \text{for } k - i = \pm 1 \text{ and/or } l - j = \pm 1 \\ 0 & \text{otherwise} \end{cases}. \quad (12)$$

For problems in which the boundary values are 0, Eq. 3 then can be used to find all of the interior amplitudes ψ_{ij} .

In order to use this technique to solve the boundary value problem discussed in Lecture Notes #5, we have to make one modification. The boundary value of $\Phi(x, a) = V_0$ is not consistent with the derivation of Eq. (4), however, since we are only interested in the region $0 \leq y \leq a$, we can extend our numerical analysis to the region $0 \leq y \leq a + h$ and require $\Phi(x, a + h) = 0$ in addition to $\Phi(x, a) = V_0$. Using the same indexing as in Lecture Notes #5, this means that $\psi_1 = \psi_2 = \psi_3 = V_0$. The finite element approach for this problem thus can be put into the matrix form for analysis by Maple:

$$\begin{pmatrix} 8/3 & -1/3 & -1/3 & -1/3 & 0 & 0 \\ -2/3 & 8/3 & -2/3 & -1/3 & 0 & 0 \\ -1/3 & -1/3 & 8/3 & -1/3 & -1/3 & -1/3 \\ -2/3 & -1/3 & -2/3 & 8/3 & -2/3 & -1/3 \\ 0 & 0 & -1/3 & -1/3 & 8/3 & -1/3 \\ 0 & 0 & -2/3 & -1/3 & -2/3 & 8/3 \end{pmatrix} \begin{pmatrix} \psi_5 \\ \psi_6 \\ \psi_8 \\ \psi_9 \\ \psi_{11} \\ \psi_{12} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} V_0. \quad (13)$$

The solution to these equations and the exact results are found to be:

$$\begin{pmatrix} \psi_5 \\ \psi_6 \\ \psi_8 \\ \psi_9 \\ \psi_{11} \\ \psi_{12} \end{pmatrix} = \begin{pmatrix} 0.5070276498 \\ .5847926267 \\ 0.1928571429 \\ 0.2785714286 \\ 0.07154377880 \\ 0.1009216590 \end{pmatrix} V_0; \quad (\text{exact}) = \begin{pmatrix} .4320283318 \\ .5405292183 \\ .1820283318 \\ 0.25 \\ .06797166807 \\ .09541411792 \end{pmatrix} V_0. \quad (14)$$

We see that the results are similar to those obtained using the finite difference approach.