Notes for Lecture #7

Orthogonal function expansions and Green's functions

Suppose we have a "complete" set of orthogonal functions $\{u_n(x)\}$ defined in the interval $x_1 \leq x \leq x_2$ such that

$$\int_{x_1}^{x_2} u_n(x)u_m(x)\, dx = \delta_{nm}. \tag{1}$$

We showed that the completeness of this functions implies that

$$\sum_{n=1}^{\infty} u_n(x)u_n(x') = \delta(x - x'). \tag{2}$$

This relation allows us to use these functions to represent a Green's function for our system. For the 1-dimensional Poisson equation, the Green's function satisfies

$$\frac{\partial^2}{\partial x^2} G(x, x') = -4\pi\delta(x - x'). \tag{3}$$

Therefore, if

$$\frac{d^2}{dx^2} u_n(x) = -\alpha_n u_n(x), \tag{4}$$

where $\{u_n(x)\}$ also satisfy the appropriate boundary conditions, then we can write the Greens functions as

$$G(x, x') = 4\pi \sum_n \frac{u_n(x)u_n(x')}{\alpha_n}. \tag{5}$$

For example, if $u_n(x) = \sqrt{2/a}\sin(n\pi x/a)$, then

$$G(x, x') = \frac{8\pi}{a} \sum_n \frac{\sin(n\pi x/a)\sin(n\pi x'/a)}{\left(\frac{n\pi}{a}\right)^2}. \tag{6}$$

These ideas can easily be extended to two and three dimensions. For example if $\{u_n(x)\}$, $\{v_n(x)\}$, and $\{w_n(x)\}$ denote the complete functions in the x, y, and z directions respectively, then the three dimensional Green's function can be written:

$$G(x, x', y, y', z, z') = \sum_{lmn} \frac{u_l(x)u_l(x')v_m(y)v_m(y')w_n(z)w_n(z')}{\alpha_l + \beta_m + \gamma_n}, \tag{7}$$

where

$$\frac{d^2}{dx^2} u_l(x) = -\alpha_l u_l(x), \quad \frac{d^2}{dy^2} v_m(x) = -\beta_m v_m(y), \quad \text{and} \quad \frac{d^2}{dz^2} w_n(z) = -\gamma_n w_n(z). \tag{8}$$

See Eq. 3.167 in **Jackson** for an example.

An alternative method of finding Green's functions for second order ordinary differential equations is based on a product of two independent solutions of the homogeneous equation, $u_1(x)$ and $u_2(x)$, which satisfy the boundary conditions at x_1 and x_1, respectively:

$$G(x,x') = K u_1(x_<) u_2(x_>), \quad \text{where} \quad K \equiv \frac{4\pi}{u_1 \frac{du_2}{dx} - \frac{du_1}{dx} u_2}, \tag{9}$$

with $x_<$ meaning the smaller of x and x' and $x_>$ meaning the larger of x and x'. For example, we have previously discussed the example of the one dimensional Poisson equation with the boundary condition $\Phi(0) = 0$ and $\frac{d\Phi(a)}{dx} = 0$ to have the form:

$$G(x,x') = -4\pi x_<. \tag{10}$$

For the two and three dimensional cases, we can use this technique in one of the dimensions in order to reduce the number of summation terms. These ideas are discussed in Section 3.11 of **Jackson**. For the two dimensional case, for example, we can assume that the Green's function can be written in the form:

$$G(x,x',y,y') = \sum_n u_n(x) u_n(x') g_n(y,y'). \tag{11}$$

We require that G satisfy the equation:

$$\nabla^2 G = \sum_n u_n(x) u_n(x') \left[-\alpha_n + \frac{\partial^2}{\partial y^2} \right] g_n(y,y') = -4\pi \delta(x-x')\delta(y-y'). \tag{12}$$

This form will have the required behavior, if we choose:

$$\left[-\alpha_n + \frac{\partial^2}{\partial y^2} \right] g_n(y,y') = -4\pi \delta(y-y'). \tag{13}$$

If $v_{n_1}(y)$ and $v_{n_2}(y)$ are solutions to the homogeneous equation

$$\left[-\alpha_n + \frac{d^2}{dy^2} \right] v_{n_i}(y) = 0, \tag{14}$$

satisfying the appropriate boundary conditions, we can then construct the 2-dimensional Green's function from

$$G(x,x',y,y') = \sum_n u_n(x) u_n(x') K_n v_{n_1}(y_<) v_{n_2}(y_>), \tag{15}$$

where the constant K_n is defined in a similar way to the one-dimensional case. For example, a Green's function for a two-dimensional with $0 \leq x \leq a$ and $0 \leq y \leq b$, with the potential vanishing on each of the boundaries can be expanded:

$$G(x,x',y,y') = 8 \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi x'}{a}\right) \sinh\left(\frac{n\pi y_<}{a}\right) \sinh\left(\frac{n\pi}{a}(b-y_>)\right)}{n \sinh\left(\frac{n\pi b}{a}\right)}. \tag{16}$$