Notes for Lecture #13

Dipole and quadrupole fields

The dipole moment is defined by

$$\mathbf{p} = \int d^3r\rho(r)\mathbf{r}, \tag{1}$$

with the corresponding potential

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}, \tag{2}$$

and electrostatic field

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{3\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}})-\mathbf{p}}{r^3} - \frac{4\pi}{3}\mathbf{p}\delta^3(\mathbf{r})\right\}. \tag{3}$$

The last term of the field expression follows from the following derivation. We note that Eq. (3) is poorly defined as $r \to 0$, and consider the value of a small integral of $\mathbf{E}(\mathbf{r})$ about zero. (For this purpose, we are supposing that the dipole $\mathbf{p}$ is located at $\mathbf{r} = \mathbf{0}$.) In this case we will approximate

$$\mathbf{E}(\mathbf{r} \approx \mathbf{0}) \approx \left(\int_{\text{sphere}} \mathbf{E}(\mathbf{r})\mathbf{d^3r}\right)\delta^3(\mathbf{r}). \tag{4}$$

First we note that

$$\int_{r\leq R}\mathbf{E}(\mathbf{r})\mathbf{d^3r} = -\mathbf{R^2}\int_{\mathbf{r}=\mathbf{R}}\mathbf{\Phi}(\mathbf{r})\hat{\mathbf{r}}\mathbf{d\Omega}. \tag{5}$$

This result follows from the Divergence theorm:

$$\int_{\text{vol}}\nabla\cdot\mathcal{V}\mathbf{d^3r} = \int_{\text{surface}}\mathcal{V}\cdot\mathbf{dA}. \tag{6}$$

In our case, this theorem can be used to prove Eq. (5) for each cartesian coordinate if we choose $\mathcal{V} \equiv \hat{\mathbf{x}}\mathbf{\Phi}(\mathbf{r})$ for the $x-$ component for example:

$$\int_{r\leq R}\nabla\Phi(\mathbf{r}) = \hat{\mathbf{x}}\int_{\mathbf{r}\leq\mathbf{R}}\nabla\cdot(\hat{\mathbf{x}}\mathbf{\Phi})\mathbf{d^3r} + \hat{\mathbf{y}}\int_{\mathbf{r}\leq\mathbf{R}}\nabla\cdot(\hat{\mathbf{y}}\mathbf{\Phi})\mathbf{d^3r} + \hat{\mathbf{z}}\int_{\mathbf{r}\leq\mathbf{R}}\nabla\cdot(\hat{\mathbf{z}}\mathbf{\Phi})\mathbf{d^3r}, \tag{7}$$

which is equal to

$$\int_{r=R}\Phi(\mathbf{r})R^2d\Omega\left((\hat{\mathbf{x}}\cdot\hat{\mathbf{r}})\hat{\mathbf{x}} + (\hat{\mathbf{y}}\cdot\hat{\mathbf{r}})\hat{\mathbf{y}} + (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})\hat{\mathbf{z}}\right) = \int_{r=R}\Phi(\mathbf{r})R^2d\Omega\hat{\mathbf{r}}. \tag{8}$$

Thus,

$$\int_{r\leq R}\mathbf{E}(\mathbf{r})\mathbf{d^3r} = -\int_{\mathbf{r}\leq\mathbf{R}}\nabla\mathbf{\Phi}(\mathbf{r})\mathbf{d^3r} = -\mathbf{R^2}\int_{\mathbf{r}=\mathbf{R}}\mathbf{\Phi}(\mathbf{r})\hat{\mathbf{r}}\mathbf{d\Omega}. \tag{9}$$

Now, we notice that the electrostatic potential can be determined from the charge density $\rho(\mathbf{r})$ according to:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi\epsilon_0}\sum_{lm}\frac{4\pi}{2l+1}\int d^3r'\rho(\mathbf{r}')\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\hat{\mathbf{r}})Y_{lm}(\hat{\mathbf{r}'}). \tag{10}$$

We also note that the unit vector can be written in terms of spherical harmonic functions:

$$\hat{\mathbf{r}} = \begin{cases} \sin(\theta)\cos(\phi)\hat{\mathbf{x}} + \sin(\theta)\sin(\phi)\hat{\mathbf{y}} + \cos(\theta)\hat{\mathbf{z}} \\ \sqrt{\frac{4\pi}{3}}\left(Y_{1-1}(\hat{\mathbf{r}})\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}})\frac{\hat{\mathbf{x}}-\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}})\hat{\mathbf{z}}\right) \end{cases} \tag{11}$$

Therefore, when we evaluate the integral over solid angle Ω in Eq. (5), only the $l = 1$ term contributes and the effect of the integration reduced to the expression:

$$-R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}\mathbf{d\Omega} = -\frac{\mathbf{1}}{\mathbf{4}\pi\epsilon_\mathbf{0}}\frac{\mathbf{4}\pi\mathbf{R^2}}{\mathbf{3}}\int \mathbf{d^3r}'\rho(\mathbf{r}')\frac{\mathbf{r}_<}{\mathbf{r}_>^\mathbf{2}}\hat{\mathbf{r}'}. \tag{12}$$

The choice of $r_<$ and $r_>$ is a choice between the integration variable r' and the sphere radius R. If the sphere encloses the charge distribution $\rho(\mathbf{r}')$, then $r_< = r'$ and $r_> = R$ so that Eq. (12) becomes

$$-R^2\int_{r=R}\Phi(\mathbf{r})\hat{\mathbf{r}}\mathbf{d\Omega} = -\frac{\mathbf{1}}{\mathbf{4}\pi\epsilon_\mathbf{0}}\frac{\mathbf{4}\pi\mathbf{R^2}}{\mathbf{3}}\frac{\mathbf{1}}{\mathbf{R^2}}\int \mathbf{d^3r}'\rho(\mathbf{r}')\mathbf{r}'\hat{\mathbf{r}'} \equiv -\frac{\mathbf{p}}{\mathbf{3}\varepsilon_\mathbf{0}}. \tag{13}$$