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PHY 711 – Notes on Hydrodynamics – ("Solitary Waves"[1])

Basic assumptions

We assume that we have in incompressible fluid (ρ = constant) a velocity potential of the form $\Phi(x, z, t)$, where

$$\mathbf{v}(x, z, t) = -\nabla\Phi(x, z, t). \tag{1}$$

The surface of the fluid is described by $h + \zeta(x, t) = z$. The fluid is contained in a tank with a structureless bottom (defined by the plane $z = 0$) and is filled to a vertical height h at equilibrium. These functions satisfy the following conditions.
The continuity equation ($\nabla \cdot \mathbf{v} = 0$) becomes the Laplace equation for $\Phi(x, z, t)$:

$$\frac{\partial^2\Phi(x, z, t)}{\partial x^2} + \frac{\partial^2\Phi(x, z, t)}{\partial z^2} = 0 \tag{2}$$

If we assume irrotational flow, $\nabla \times \mathbf{v} = 0$, we also have the Bernoulli equation in the form:

$$-\frac{\partial\Phi(x, z, t)}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\Phi(x, z, t)}{\partial x}\right)^2 + \left(\frac{\partial\Phi(x, z, t)}{\partial z}\right)^2\right] + g(z - h) = 0. \tag{3}$$

Here we have assumed that the potential energy is due to gravity and have taken the reference potential energy at the height $z = h$. The boundary conditions for this system take the form of zero vertical velocity at bottom of the tank:

$$\frac{\partial\Phi(x, 0, t)}{\partial z} = 0. \tag{4}$$

At the surface of the fluid, $z = h + \zeta(x, t)$ we expect that

$$v_z(x, z, t)\rfloor_{z=h+\zeta} = \frac{d\zeta}{dt} = \mathbf{v} \cdot \nabla\zeta + \frac{\partial\zeta}{\partial t}. \tag{5}$$

This becomes:

$$-\frac{\partial\Phi(x, z, t)}{\partial z} + \frac{\partial\Phi(x, z, t)}{\partial x}\frac{\partial\zeta(x, t)}{\partial x} - \frac{\partial\zeta(x, t)}{\partial t}\Bigg\rfloor_{z=h+\zeta} = 0 \tag{6}$$

In this treatment, we assume seek the form of surface waves traveling along the $x-$ direction and assume that the effective wavelength is much larger than the height of the surface h. This allows us to approximate the $z-$ dependence of $\Phi(x, z, t)$ by means of a Taylor series expansion:

$$\Phi(x, z, t) \approx \Phi(x, 0, t) + z\frac{\partial\Phi}{\partial z}(x, 0, t) + \frac{z^2}{2}\frac{\partial^2\Phi}{\partial z^2}(x, 0, t) + \frac{z^3}{3!}\frac{\partial^3\Phi}{\partial z^3}(x, 0, t) + \frac{z^4}{4!}\frac{\partial^4\Phi}{\partial z^4}(x, 0, t)\cdots \tag{7}$$

This expansion can be simplified because of the bottom boundary condition (4) which ensures that all odd derivatives $\frac{\partial^n\Phi}{\partial z^n}(x, 0, t)$ vanish from the Taylor expansion (7). In addition, the Poisson equation (2) allows us to convert all even derivatives with respect to z to derivatives with respect to x. Therefore, the expansion (7) becomes:

$$\Phi(x, z, t) \approx \Phi(x, 0, t) - \frac{z^2}{2}\frac{\partial^2\Phi}{\partial x^2}(x, 0, t) + \frac{z^4}{4!}\frac{\partial^4\Phi}{\partial x^4}(x, 0, t)\cdots \tag{8}$$

Before seeking the form of the nonlinear equations, we first consider the linearized version of these equations. We focus on the solution at the free surface $z = h + \zeta(x,t)$. The linear version of the Bernoulli equation evaluated at the free surface is

$$-\frac{\partial \Phi(x,h,t)}{\partial t} + g\zeta(x,t) = 0. \tag{9}$$

The linearized surface boundary condition is

$$-\frac{\partial \Phi(x,z,t)}{\partial z} - \frac{\partial \zeta(x,t)}{\partial t}\bigg|_{z=h+\zeta} = 0 \tag{10}$$

Using the Taylor's expansion in this surface boundary

$$-\frac{\partial \Phi(x,z,t)}{\partial z} \approx h\frac{\partial^2 \Phi(x,0,t)}{\partial x^2} = \frac{\partial \zeta(x,t)}{\partial t}. \tag{11}$$

Eliminating ζ from the coupled Eqs. (9) and (11), we find

$$\frac{\partial^2 \Phi(x,0,t)}{\partial t^2} = gh\frac{\partial^2 \Phi(x,0,t)}{\partial x^2}, \tag{12}$$

a wave equation with velocity $c = \sqrt{gh}$.

We now return to treating the nonlinear equations.
For convenience we define $\phi(x,t) \equiv \Phi(x,0,t)$. Using Eq. (8), the Bernoulli equation (3) then becomes:

$$-\frac{\partial \phi}{\partial t} + \frac{(h+\zeta)^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left((h+\zeta)\frac{\partial^2 \phi}{\partial x^2}\right)^2\right] + g\zeta = 0, \tag{13}$$

where we have discarded some of the higher order terms. Keeping all terms up to leading order in non-linearity and up to fourth order derivatives in the linear terms, the Bernoulli equation becomes:

$$-\frac{\partial \phi}{\partial t} + \frac{h^2}{2}\frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + g\zeta = 0. \tag{14}$$

Using a similar analysis and approximation, the surface definition equation (6) becomes:

$$\frac{\partial}{\partial x}\left((h + \zeta(x,t))\frac{\partial \phi}{\partial x}\right) - \frac{h^3}{3!}\frac{\partial^4 \phi}{\partial x^4} - \frac{\partial \zeta}{\partial t} = 0, \tag{15}$$

We would like to solve Eqs. (14-15) for a traveling wave of the form:

$$\phi(x,t) = \chi(x - ct) \text{ and } \zeta(x,t) = \eta(x - ct), \tag{16}$$

where the speed of the wave c will be determined. Letting $u \equiv x - ct$, Eqs. (14 and 15) become:

$$\frac{d}{du}\left((h + \eta(u))\frac{d\chi(u)}{du}\right) - \frac{h^3}{6}\frac{d^4\chi(u)}{du^4} + c\frac{d\eta(u)}{du} = 0, \tag{17}$$

and

$$c\frac{d\chi(u)}{du} - \frac{ch^2}{2}\frac{d^3\chi(u)}{du^3} + \frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2 + g\eta(u) = 0. \tag{18}$$

The modified surface equation (17) can be integrated once with respect to u, choosing the constant of integration to be zero and giving the new form for the surface condition:

$$(h+\eta)\chi' - \frac{h^3}{6}\chi''' + c\eta = 0, \tag{19}$$

where we have abreviated derivatives with respect to u with the "$'$" symbol. This equation, and the modified Bernoulli equation (14) are now two coupled non-linear equations. In order to solve them, we use, the modified Bernoulli equation to approximate $\chi'(u)$ and its higher derivatives in terms of the surface function $\eta(u)$. Equation (14) becomes approximately:

$$\chi' = -\frac{g}{c}\eta + \frac{h^2}{2}\chi''' - \frac{1}{2c}(\chi')^2 \approx -\frac{g}{c}\eta - \frac{h^2 g}{2c}\eta'' - \frac{g^2}{2c^3}\eta^2. \tag{20}$$

Using similar approximations, we can eliminate $\chi'(u)$ and its higher derivatives from the surface equation (19):

$$(h+\eta)\left(-\frac{g}{c}\eta - \frac{h^2 g}{2c}\eta'' - \frac{g^2}{2c^3}\eta^2\right) + \frac{h^3 g}{6c}\eta'' + c\eta = 0, \tag{21}$$

where some terms involving non-linearity of higher than 2 or involving higher order derivatives have been discarded. Collecting the leading terms, we obtain:

$$\left(1 - \frac{gh}{c^2}\right)\eta - \frac{gh^3}{3c^2}\eta'' - \frac{g}{c^2}\left(1 + \frac{gh}{2c^2}\right)\eta^2 = 0. \tag{22}$$

For the second two terms, *Fetter and Walecka* argue that it is consistent to approximate $gh \approx c^2$, which reduces (22) to

$$\left(1 - \frac{hg}{c^2}\right)\eta(u) - \frac{h^2}{3}\eta''(u) - \frac{3}{2h}\left[\eta(u)\right]^2 = 0. \tag{23}$$

Your text shows that a solution to Eq. (23) (corresponding to Eq. 56.30 of the text), with the initial condition $\eta(0) = \eta_0$ and $\eta'(0) = 0$, is the solitary wave form:

$$\zeta(x,t) = \eta(x-ct) = \eta_0 \operatorname{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right), \tag{24}$$

with

$$c = \sqrt{\frac{gh}{1-\eta_0/h}} \approx \sqrt{gh}\left(1 + \frac{\eta_0}{2h}\right). \tag{25}$$

The "standard" form of the related Korteweg-de Vries equation[2] is given in terms of the scaled variables $\bar{t}$ and $\bar{x}$ in terms of the function $\eta(\bar{x},\bar{t})$ by

$$\frac{\partial \eta}{\partial \bar{t}} + 6\eta\frac{\partial \eta}{\partial \bar{x}} + \frac{\partial^3 \eta}{\partial \bar{x}^3} = 0, \tag{26}$$

which has a solution

$$\eta(\bar{x},\bar{t}) = \frac{\beta}{2}\operatorname{sech}^2\left[\frac{\sqrt{\beta}}{2}(\bar{x}-\beta\bar{t})\right]. \tag{27}$$

This form is related to our results in the following way.

$$\beta = 2\eta_0, \qquad \bar{x} = \sqrt{\frac{3}{2h}}\frac{x}{h}, \qquad \text{and} \qquad \bar{t} = \sqrt{\frac{3}{2h}}\frac{ct}{2\eta_0 h}. \tag{28}$$

To show how the reduced equation (23) is related to the Korteweg-de Vries equation, we first take the u derivative to find:

$$\frac{\eta_0}{h}\eta' - \frac{h^2}{3}\eta''' - \frac{3}{h}\eta\eta' = 0, \tag{29}$$

where we have used the relation

$$\frac{\eta_0}{h} = 1 - \frac{gh}{c^2}. \tag{30}$$

Then we notice that

$$\frac{\partial \eta}{\partial t} = -c\frac{d\eta}{du} \quad \text{and} \quad \frac{\partial \eta}{\partial x} = \frac{d\eta}{du}, \tag{31}$$

so that Eq. (29) can be written:

$$-\frac{\eta_0}{ch}\frac{\partial \eta}{\partial t} - \frac{h^2}{3}\frac{\partial^3 \eta}{\partial x^3} - \frac{3}{h}\eta\frac{\partial \eta}{\partial x} = 0. \tag{32}$$

Substituting the transformation (28) into this partial differential equation yields the Korteweg-de Vries equation (26).

References

[1] Alexander L. Fetter and John Dirk Walecka, **Theoretical Mechanics of Particles and Continua**, (McGraw Hill, 1980), Chapt. 10.

[2] Websites concerning solitons:

http://www.ma.hw.ac.uk/solitons/,

http://www.usf.uni-osnabrueck.de/∼kbrauer/solitons.html,

http://www.math.h.kyoto-u.ac.jp/∼takasaki/soliton-lab/gallery/index-e.html