September 10, 1999

PHY 711 – Lecture notes on Lagrangian for Electric and Magnetic Fields

For simplicity, consider a Lagrangian for a single particle having the form (in Cartesian coordinates) $L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$. The Euler-Lagrange equations have the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \tag{1}$$

with similar equations for y and z. We can show that this form is consistent with Newton's Laws if the potential function U takes the form:

$$U = U^0(x, y, z, t) + U^{EM}(x, y, z, \dot{x}, \dot{y}, \dot{z}, t), \tag{2}$$

where U^{EM} represents the interaction of our particle (having charge q) with an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ where we can represent the fields in terms of the scalar and vector potentials:

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{3}$$

We must find U^{EM} which is both consistent with the Euler-Lagrange Eq.(1) and with the Lorentz force (written in the x direction):

$$F_x = q(E_x + \frac{1}{c}(\dot{\mathbf{r}} \times \mathbf{B})\rfloor_x) = -\frac{\partial U^{EM}}{\partial x} + \frac{d}{dt}\left(\frac{\partial U^{EM}}{\partial \dot{x}}\right). \tag{4}$$

We note that the magnetic field terms can be evaluated:

$$\dot{\mathbf{r}} \times (\nabla \times \mathbf{A})\rfloor_x = \dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right). \tag{5}$$

The right hand side of Eq.(5) (with the addition and subtraction of a convenient term) can be written:

$$\underbrace{\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}}_{\partial(\dot{\mathbf{r}}\cdot\mathbf{A})/\partial x} - \underbrace{\dot{x}\frac{\partial A_x}{\partial x} - \dot{y}\frac{\partial A_x}{\partial y} - \dot{z}\frac{\partial A_x}{\partial z}}_{-dA_x/dt + \partial A_x/\partial t}, \tag{6}$$

where we are assuming that $A_x = A_x(x, y, z, t)$. Noting that $A_x = \partial(\dot{\mathbf{r}} \cdot \mathbf{A})/\partial \dot{x}$, the electromagnetic force can thus be written:

$$F_x = \underbrace{-q\frac{\partial \phi}{\partial x} - \frac{q}{c}\frac{\partial A_x}{\partial t}}_{qE_x} + \frac{q}{c}\left(\underbrace{\frac{\partial(\dot{\mathbf{r}} \cdot \mathbf{A})}{\partial x} - \frac{d}{dt}\frac{\partial(\dot{\mathbf{r}} \cdot \mathbf{A})}{\partial \dot{x}} + \frac{\partial A_x}{\partial t}}_{(\dot{\mathbf{r}}\times\mathbf{B})\rfloor_x}\right). \tag{7}$$

Simplifying this equation, we obtain

$$F_x = -\frac{\partial}{\partial x}\left(q\phi - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}\right) - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left(\frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}\right). \tag{8}$$

Thus, we finally have the result

$$U^{EM} = q\phi - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}. \tag{9}$$