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Winning probabilities in a pairwise lottery system with three alternatives*

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Summary. The pairwise lottery system is a multiple round voting procedure which chooses by lot a winner from a pair of alternatives to advance to the next round where in each round the odds of selection are based on each alternative's majority rule votes. We develop a framework for determining the asymptotic relative likelihood of the lottery selecting in the final round the Borda winner, Condorcet winner, and Condorcet loser for the three alternative case. We also show the procedure is equivalent to a Borda lottery when only a single round of voting is conducted. Finally, we present an alternative voting rule which yields the same winning probabilities as the pairwise lottery in the limiting case as the number of rounds of the pairwise lottery becomes large.

Keywords and Phrases: Probabilistic voting, Lottery systems, Markov chain, Majority Rule, Borda Rule, Condorcet Rule.

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1 Introduction

Majority rule is a standard procedure often utilized to decide from a pair of alternatives. When there is an exact tie, a coin-flip is generally advocated to determine the winner. A coin flip is considered fair in this case because each alternative received an equal number of votes. Proportional lottery rules generalize this principle to choose the winner based upon the percentage of votes each alternative receives. When there are three alternatives under consideration, there are a variety of ways to tally the initial votes. The simplest mechanism is based on the plurality rule.

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Zeckhauser (1973) shows a plurality lottery is akin to appointing a randomly chosen voter as the dictator. Gehrlein (1991) calculates the likelihood this proportional lottery will select the majority rule winner. Recently, the properties of a Borda rule lottery have been developed by Heckelman (2003). Under any of these single round lottery systems, the alternative with the most points, no matter how the points are initially tallied, is always more likely to win compared to any other alternative.

Alternatively, Mueller (1989) advocates a series of pairwise vote lotteries with the winner determined by the lottery in an arbitrarily determined final round. Winning probabilities under Mueller's multiple round procedure are less straightforward to determine than for the single round lotteries. We develop a general framework to examine Mueller's pairwise lottery system in more detail. We are able to determine the convergent probabilities that each alternative will be selected the ultimate winner and compare against alternatives that have certain properties. For example, we discover that a Condorcet winner, defined as an alternative that wins a majority rule vote head to head against all other alternatives, does not necessarily have the highest probability of being chosen under the pairwise lottery. At the other extreme, if only a single round of voting occurs, it turns out the winning probabilities for each alternative are identical to the Borda-weighted lottery. Finally, we offer an alternative lottery system which has the same convergent probabilities as the pairwise lottery for selecting each alternative but also has the advantage of containing an endogenous termination rule rather than relying on an arbitrary stopping point as in the pairwise lottery system.

2 Details of the procedure

There are three alternatives, A, B, and C, from which a group must choose. Let $N_{(x,y,z)}$ denote the proportion of voters whose preference ranking of alternatives from most preferred to least preferred is given by (x, y, z), where $x, y, z \in \{A, B, C\}$ and $x \neq y \neq z$. The proportion of voters who prefer alternative x to alternative y is therefore given by

$$P_{xy} = N_{(x,y,z)} + N_{(x,z,y)} + N_{(z,x,y)}.$$

Note that $P_{xy} + P_{yx} = 1$.

Mueller's pairwise lottery (PWL) procedure is as follows: two alternatives are selected at random from a uniform distribution and voting is conducted. A lottery is then held with each of the two alternatives being selected with probability equal to the proportion of votes it received. The winning alternative then faces the remaining alternative and voting is conducted again, with the new lottery probabilities equal to the proportion of votes each alternative received. The selected alternative then faces the losing alternative from the previous vote, and a new lottery is held. The steps repeat for an undisclosed period of time. In each pairwise lottery contest between alternative x and alternative y, P_{xy} is the probability that alternative x is selected over alternative y.

As Mueller points out, strategic voting can alter winning probabilities and future pairings but he dismisses such behavior as being very unlikely to occur due to the difficulty of correctly identifying the optimal strategy. We assume sincere voting in each round throughout the rest of the paper. Voters cannot determine at the start which alternatives will meet in the final round, but they can compute the expected probabilities, and therefore the probability of each alternative being selected the ultimate winner. In attempting to calculate these probabilities, Mueller makes the mistake of assuming an equal probability for each potential vote pair to be in the final round, but this assumption only holds true in the first round. After that, future round pairings are a function of P_{xy} for all x, y.[1]

In order to derive the winning probabilities for each candidate from the PWL procedure, we make use of the fact that the probabilities $\{P_{xy}\}$ define a Markov process over all possible pairs (x, y). Specifically, note that, given any current pair (x, y), the probability that the pair $(x, z) \neq (x, y)$ will be chosen in the next round, which can occur if and only if x is selected over y, is P_{xy}. Hence, the distribution of pairs that are selected in each round of the voting procedure can be determined using the Markov transition matrix

$$\boldsymbol{\Pi} = \begin{array}{c|c|c|c|}
 & (A,B) & (A,C) & (B,C) \\ \cline{2-4}
(A,B) & 0 & P_{AB} & P_{BA} \\ \cline{2-4}
(A,C) & P_{AC} & 0 & P_{CA} \\ \cline{2-4}
(B,C) & P_{BC} & P_{CB} & 0 \\ \cline{2-4}
\end{array}.$$

In particular, the probability that any pair (x, y) will be selected in the n-th round for any $n \geq 1$ can be obtained from the transition matrix $\boldsymbol{\Pi}^{n-1}$. (For example, the entry in row 2 and column 3 of $\boldsymbol{\Pi}^{n-1}$ gives the probability that, given the initial pairing (A, C), the pair (B, C) will be the pair selected in the n-th round.) Recall that each pair has an equal probability of being selected in the first round. Define $\boldsymbol{\Theta} = [\,\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\,]$ where each element in $\boldsymbol{\Theta}$ denotes the probabilities that any particular pair (x, y) is the initial pair selected. The probability distribution over the round-n pairing is therefore given by $\boldsymbol{\Theta\Pi}^{n-1}$, which belongs to $\triangle^3$, the positive unit simplex in $\Re^3$. Specifically, the first element of $\boldsymbol{\Theta\Pi}^{n-1}$ is the probability that the pair (A, B) is selected in round n given that the initial pairing is drawn uniformly random. Similarly, the second and third element of $\boldsymbol{\Theta\Pi}^{n-1}$ give the probability with which the pairs (A, C) and (B, C), respectively, are selected in round n.

Now, assuming that the procedure terminates after exactly n rounds, if the pair (x, y) is selected in round n, then, given this, the probability that x is chosen as the winner of the n-round PWL is P_{xy}. On the other hand, an alternative has probability 0 of winning the n-round PWL if the pair selected in the n-th round does not include

[1] To show this directly, consider Mueller's example in which $P_{AC} = P_{BA} = P_{BC} = 2/3$ (and thus $P_{CA} = P_{AB} = P_{CB} = 1/3$). A second round vote between A and B will occur only if C is selected as part of the first round paring and subsequently loses. Given that the first round pairings are selected at random, then before the first round alternatives are announced, the expected probability of C losing a first round vote is equal to $\frac{1}{3}P_{AC} + \frac{1}{3}P_{BC}$, which in this particular example is $\frac{1}{3}(\frac{2}{3} + \frac{2}{3}) = \frac{4}{9}$. Likewise, the probability of a second round vote involving A and C is equal to $\frac{1}{3}(P_{AB} + P_{CB}) = \frac{1}{3}(\frac{1}{3} + \frac{1}{3}) = \frac{2}{9}$, and the probability of a second round vote between B and C is equal to $\frac{1}{3}(P_{BA} + P_{CA}) = \frac{1}{3}(\frac{2}{3} + \frac{1}{3}) = \frac{3}{9}$. Thus, the expected probabilities for any particular pair to be involved in a future round of voting beyond the first round are not equal for all potential vote pairs.

the alternative in question. Therefore, the probability that alternative x will win the n-round PWL, which we denote by $W_{x,n}$, is

$$W_{x,n} = \boldsymbol{\Theta\Pi}^{n-1}\mathbf{P}_x, \tag{1}$$

where

$$\mathbf{P}_A = \begin{bmatrix} P_{AB} \\ P_{AC} \\ 0 \end{bmatrix}, \mathbf{P}_B = \begin{bmatrix} P_{BA} \\ 0 \\ P_{BC} \end{bmatrix}, \mathbf{P}_C = \begin{bmatrix} 0 \\ P_{CA} \\ P_{CB} \end{bmatrix}.$$

2.1 Identifying the Borda winner

Mueller had calculated expected utility for each voter, given an example of specific utilities, when each alternative had an equal probability of being one of the two alternatives in the final round. As stated above, this would only occur as a general rule when there is only a single round of voting, which corresponds to the case of $n = 1$ in (1). In this particular case,

$$W_{A,1} = \frac{1}{3}(P_{AB} + P_{AC}) \tag{2A}$$

$$W_{B,1} = \frac{1}{3}(P_{BA} + P_{BC}) \tag{2B}$$

$$W_{C,1} = \frac{1}{3}(P_{CA} + P_{CB}). \tag{2C}$$

An interesting result is that this corresponds to the probabilities under the Borda lottery developed in Heckelman (2003). Borda's rule assigns 2 points to a voter's top choice, 1 point to a voter's second choice, and 0 points to a voter's third choice. Points are then totaled across all voters and the alternative with the most points is the winner. A Borda-weighted lottery is a single draw to determine the winning alternative where each alternative is selected with odds equal to its Borda score relative to the sum of all the alternatives' Borda scores. The probability that, for example, alternative A is selected from a lottery based on Borda points is given by

$$\frac{2N_{(A,B,C)} + 2N_{(A,C,B)} + N_{(B,A,C)} + N_{(C,A,B)}}{3}$$
$$= \left[\frac{N_{(A,B,C)} + N_{(A,C,B)} + N_{(C,A,B)}}{3}\right] + \left[\frac{N_{(A,B,C)} + N_{(A,C,B)} + N_{(B,A,C)}}{3}\right]$$
$$= \frac{P_{AB}}{3} + \frac{P_{AC}}{3}$$

which is the same as (2A) above. The reason for this equivalence is that although Borda's rule is normally depicted as a positional system, this rule can also be envisioned as a pairwise voting system since the Borda points assigned by each voter to a particular alternative represent the number of other alternatives ranked below it (Levin and Nalebuff, 1995). Thus the Borda score for each alternative is equal to the number of its pairwise votes.

Given this equivalence, we are now in position to identify the Borda winner from the PWL winning probabilities. By definition, since the Borda winner has the most points, then the Borda winner is more likely to be chosen from a Borda lottery than any other specific alternative. This implies that alternative x is the Borda winner if and if only $W_{x,1} > W_{y,1}\ \forall y \neq x$.

2.2 Identifying the Condorcet winner and Condorcet loser

We can also define a Condorcet winner and Condorcet loser in our framework. The Condorcet winner, if it exists, is preferred by a majority of voters in each pairwise competition against every other alternative. A Condorcet loser, if it exists, is preferred by less than a majority in each pairwise competition against every other alternative. Thus, x is a Condorcet winner if and only if $P_{xy} > 1/2\ \forall y \neq x$ in the pairwise lottery. Similarly, z is a Condorcet loser if and only if $P_{zy} < 1/2\ \forall y \neq z$.

3 Asymptotic properties of the pairwise lottery

The PWL ends after an unspecified number of rounds. It can be shown that, given any alternative x, the sequence of winning probabilities $\{W_{x,n}\}$ converges to a unique limit as the number of rounds becomes large. The proof is presented in the appendix.

Proposition 1 *For any alternative x,*

$$\lim_{n\to\infty} W_{x,n} = \boldsymbol{\Theta}^*\mathbf{P}_x,$$

where $\boldsymbol{\Theta}^* \in \triangle^3$ is the unique invariant distribution of Π, i.e. $\boldsymbol{\Theta}^*\boldsymbol{\Pi} = \boldsymbol{\Theta}^*$.

Straightforward algebraic calculations yield

$$\begin{aligned}\boldsymbol{\Theta}^* &= \frac{1}{P_{BC}(P_{AB} - P_{AC}) + P_{AC}P_{BA} + 2} \\ &\quad \times [P_{BC} + P_{AC}\ P_{CB}P_{CB} + P_{AB}\ P_{BC}P_{CA} + P_{BA}\ P_{AC}].\end{aligned}$$

Therefore,

$$W_A = \lim_{n\to\infty} W_{A,n} = \frac{P_{AB}P_{BC} + P_{AC}P_{CB} + P_{AB}P_{AC}}{P_{BC}(P_{AB} - P_{AC}) + P_{AC}P_{BA} + 2} \tag{3A}$$

$$W_B = \lim_{n\to\infty} W_{B,n} = \frac{P_{BC}P_{CA} + P_{BA}P_{AC} + P_{BA}P_{BC}}{P_{BC}(P_{AB} - P_{AC}) + P_{AC}P_{BA} + 2} \tag{3B}$$

$$W_C = \lim_{n\to\infty} W_{C,n} = \frac{P_{CA}P_{AB} + P_{CB}P_{BA} + P_{CA}P_{CB}}{P_{BC}(P_{AB} - P_{AC}) + P_{AC}P_{BA} + 2}. \tag{3C}$$

We now turn to analyzing the probability outcomes when enough rounds have occurred to rely on the limiting distribution of the transition matrix. In general, the likelihood of any alternative winning the PWL depends on the product of its pairwise probabilities, as stated explicitly below.

Proposition 2 *For all n sufficiently large in the PWL, alternative x has the highest probability of winning, alternative y has the second highest probability of winning, and alternative z has the least chance of winning if $P_{xy}P_{xz} > P_{yx}P_{yz} > P_{zx}P_{zy}$.*

Proof. We simply note that from (3A)–(3C) $W_x \gtreqless W_y$ as $P_{xy}P_{xz} \gtreqless P_{yx}P_{yz}$, $W_x \gtreqless W_z$ as $P_{xy}P_{xz} \gtreqless P_{zx}P_{zy}$, and $W_y \gtreqless W_z$ as $P_{yx}P_{yz} \gtreqless P_{zx}P_{zy}$. □

Given the identification of the Borda winner, Condorcet winner, and Condorcet loser from the previous section, we can now determine the relative likelihood each would be selected the winner in a PWL vote. Note first that we already showed the Borda winner would have a higher probability of being selected in the first round of the PWL than any other specific alternative. This result, however, does not necessarily hold beyond the first round. For example, suppose $P_{AB} = \frac{2}{3}$, $P_{AC} = \frac{2}{5}$, and $P_{BC} = \frac{3}{4}$. Then from (2A)–(2C)

$$W_{A,1} = \frac{256}{720}, W_{B,1} = \frac{260}{720}, W_{C,1} = \frac{204}{720},$$

so alternative B is the Borda winner. However, for $n = 2$, alternative A has the highest probability of winning, since

$$W_{A,2} = \frac{272}{720}, W_{B,2} = \frac{260}{720}, W_{C,2} = \frac{188}{720}.$$

In fact, alternative A is the most likely winner under the PWL for all n sufficiently large, since from (3A)–(3C)

$$W_A = \frac{26}{70}, W_B = \frac{25}{70}, W_C = \frac{19}{70}.$$

Since the PWL represents a sequence of pairwise votes, it might be expected that the Condorcet winner, if it exists, would be the most likely alternative selected but this may not be true. For the single round case, we demonstrated above that the Borda winner is most likely to be chosen, and it is well established that the Condorcet winner may not generate the most Borda points.[2] Even under multiple rounds, however, the PWL may not give the greatest probability of winning to a Condorcet winner. This dichotomy occurs because the winning probabilities under the PWL are a function of all the pairwise comparisons, whereas Condorcet's rule ignores the pairwise comparison that does not involve the Condorcet winner. Thus, the PWL takes into account more information than Condorcet's rule. Furthermore, the size of the pairwise pluralities are important to determining the lottery probabilities. As stated above, a Condorcet winner exists if there is some alternative x such that $P_{xy} > 1/2 \ \forall y \neq x$, and no distinction is made for how many people prefer this alternative as long as it is always a majority. Under the lottery, however, the larger the majority, the greater the chance of being selected.

Suppose alternative A is a Condorcet winner holding only slight majority preference in each pairwise vote, with $P_{AB} = P_{AC} = \frac{55}{100}$. In addition, let B be a

[2] The likelihood of the Borda winner coinciding with the Condorcet winner has attracted much attention. For the three alternative case, see among others Fishburn and Gehrlein (1978), Lepelley (1995), Saari (1999) and Gehrlein and Lepelley (2001).

strong majority winner over C such that $P_{BC} = \frac{4}{5}$. For the limiting case in which $n \to \infty$, then from (3A)–(3C)

$$W_A = \frac{341}{899}, W_B = \frac{387}{899}, W_C = \frac{171}{899}$$

which reveals that the Condorcet winner is not the favored alternative under the lottery.

Similarly, a Condorcet loser, if it exists, need not be the least likely winner under the PWL. Suppose $P_{AB} = P_{AC} = \frac{1}{3}$ and $P_{BC} = \frac{1}{7}$. In this case, candidate A is the Condorcet loser, but candidate B has the lowest probability of winning for all n sufficiently large, since

$$W_A = \frac{14}{70}, W_B = \frac{13}{70}, W_C = \frac{43}{70}.$$

Since we have established that neither the Borda winner nor the Condorcet winner is necessarily the most likely alternative to be selected under the PWL, it is natural to ask if either the Borda or Condorcet winner will always be more likely than the other to be chosen in the limit for every possible set of voter profiles. As it turns out, however, examples can be constructed in which either the Borda winner or the Condorcet winner has the higher asymptotic winning probability.

First suppose $P_{AB} = \frac{2}{5}$, $P_{AC} = \frac{17}{20}$, and $P_{BC} = \frac{3}{5}$, so that alternative B is the Condorcet winner. In this case,

$$W_{A,1} = \frac{25}{60}, W_{B,1} = \frac{24}{60}, W_{C,1} = \frac{11}{60},$$

so alternative A is the Borda winner. For all n sufficiently large, alternative B has a higher probability of winning compared with alternative A, since

$$W_A = \frac{23}{56}, W_B = \frac{24}{56}, W_C = \frac{9}{56}.$$

Suppose instead P_{AC} is increased to $\frac{19}{20}$, with P_{AB} and P_{BC} unchanged. Alternative B remains the Condorcet winner, and

$$W_{A,1} = \frac{9}{20}, W_{B,1} = \frac{8}{20}, W_{C,1} = \frac{3}{20},$$

so alternative A is still the Borda winner. In this case, alternative A has the higher probability of winning for all n sufficiently large, since

$$W_A = \frac{25}{56}, W_B = \frac{24}{56}, W_C = \frac{7}{56}.$$

Definitive results can be obtained, however, when comparing each against the Condorcet loser.

Proposition 3 *The Condorcet winner has a higher probability of winning the PWL than the Condorcet loser for all n sufficiently large.*

Proof. From Proposition 2, $W_A \gtreqless W_B$ as $P_{AB}P_{AC} \gtreqless P_{BC}P_{BA}$. Suppose alternative A is a Condorcet winner and alternative B is a Condorcet loser. Thus $P_{AB} > \frac{1}{2}$, $P_{AC} > \frac{1}{2}$, and $P_{BC} < \frac{1}{2}$. This specification gives $P_{AB}P_{AC} > \frac{1}{4} > P_{BC}P_{BA}$, which yields $W_A > W_B$. □

Proposition 4 *The Borda winner has a higher probability of winning the PWL than the Condorcet loser for all n sufficiently large.*

Proof. Here we present a proof by contradiction. From Proposition 2, $W_A \gtreqless W_B$ as $P_{AB}P_{AC} \gtreqless P_{BC}P_{BA}$. Suppose alternative B is a Condorcet loser, so that $P_{AB} > \frac{1}{2}$ and $P_{BC} < \frac{1}{2}$. Suppose further that alternative A is a Borda winner, in which case it must be true that $W_{A,1} > \max\{W_{B,1}, W_{C,1}\}$.

Now suppose that contrary to the proposition, $W_A \leq W_B$, or, equivalently,

$$P_{AC} \leq \frac{P_{BC}P_{BA}}{P_{AB}}.$$

Given that alternative A is a Borda winner, it must be the case that $W_{A,1} - W_{C,1} = \frac{1}{3}(P_{AB} + P_{AC} - (P_{CA} + P_{CB})) > 0$. However, since $W_A \leq W_B$, then

$$\begin{aligned}
&P_{AB} + P_{AC} - (P_{CA} + P_{CB}) \\
&\leq P_{AB} + \left(\frac{P_{BC}P_{BA}}{P_{AB}}\right) - \left(1 - \left(\frac{P_{BC}P_{BA}}{P_{AB}}\right)\right) - P_{CB} \\
&= \frac{(2 - P_{AB})\,(P_{BC} - P_{AB})}{P_{AB}} < 0,
\end{aligned}$$

which is a contradiction. □

4 An equivalent procedure

Most of our definitive results are based on the limiting probabilities which require a large number of voting rounds which is cumbersome and time consuming to implement in practice. Furthermore, the arbitrariness of the final round determination in the PWL is not particularly satisfying. The same limiting winning probabilities from the PWL, however, can be attained by a different social choice rule which incorporates an endogenous termination point. This new rule is based on dual pairwise lotteries (DPWL) and may end after only a single round of dual pair voting.

The DPWL requires two different pairs of alternatives to be considered at the same time, which with only three alternatives requires one element to be common to both pairs. From the three potential sets of different pairings, one set is randomly drawn with uniform distribution. The winner from each pair is determined by proportional lottery.[3] To be selected as the final social choice, the alternative common to both pairs must win both lotteries. If this alternative only wins a single lottery,

[3] Since there are only two alternatives being considered at a time, majority rule, plurality, and Borda weights are all identical.

then the winner from the other pairing is selected as the social choice by transitivity.[4] Otherwise, if the alternative common to both pairings loses both lotteries, then the entire procedure is repeated[5] ignoring the earlier lottery results.

Therefore, in any round of the DPWL, the probability that alternative x is chosen as the social choice is

$$\frac{1}{3}(P_{xy}P_{xz} + P_{xy}P_{yz} + P_{xz}P_{zy}).$$

The probability that no winner is determined in that round, thereby requiring another set of pairs to be selected, is

$$\delta = \frac{1}{3}(P_{BA}P_{CA} + P_{AB}P_{CB} + P_{AC}P_{BC}) \leq \frac{1}{3}(P_{CA} + 1 + P_{AC}) = \frac{2}{3}.$$

Consequently, the probability that alternative x wins the DPWL is given by

$$\begin{aligned} &\frac{1}{3}\sum_{i=0}^{\infty}\delta^i(P_{xy}P_{xz} + P_{xy}P_{yz} + P_{xz}P_{zy}) \\ &= \frac{P_{xy}P_{xz} + P_{xy}P_{yz} + P_{xz}P_{zy}}{3(1-\delta)} \\ &= W_x. \end{aligned}$$

Thus if one favors the PWL as a social choice rule, the same expected outcomes from a large number of rounds can be achieved much quicker in expectation under the DPWL system. The probability that a winner is chosen in round n of the DPWL is $(1-\delta)\delta^{n-1}$. The expected number of rounds of dual paired votes needed to determine a winner using this procedure is thus equal to

$$(1-\delta)\sum_{n=1}^{\infty} n\delta^{n-1} = \frac{1}{1-\delta} \leq 3.$$

5 Future directions

Depending on the specific voter preference profiles, it was shown through examples that it is possible for either the Condorcet or Borda winner to have the greater relative probability of being selected in the PWL. Although either could conceivably be favored in the lottery process, which is more likely to be favored in general could be estimated via simulations and appropriate assumptions regarding the distribution of voter preferences. Typical examples might include if all voters were drawn from a uniform distribution (Van Newenhizen, 1992), were required to have single-peaked preferences (Lepelley, 1995), or each set of profiles was restricted to obey the Impartial Culture condition (Merlin et al., 2002). Such approaches are often taken

[4] In practice, transitivity may not hold but the decision rule dictates a transitive outcome simply in order to yield the desired outcome of matching the PWL aymptotic winning probabilities for each alternative.

[5] In this case, transitivity cannot be applied to decide between the other two alternatives.

to estimate the likelihood various procedures will select the Condorcet winner. By comparing the relative Condorcet and Borda efficiencies under the PWL for different voter preference distributions, it could be estimated which winner is more likely to be selected in the lottery.

Our results for the PWL are based on the transition matrix for three alternatives. The framework can be expanded to include any finite number of alternatives but with more than three alternatives from which to select, it is not clear how future pairings will be set. One method would be to have the winner from each round face a randomly selected opponent chosen from the set of remaining alternatives without replacement so that a losing alternative cannot be reselected for the next round until every alternative has been randomly selected for inclusion in the voting procedure an equal number of times. This can be achieved either by randomly creating a single ordering to be repeated each time all alternatives have been exhausted, or a new ordering created each time. Alternatively, replacement may be allowed after each lottery. Results may depend on the specific method utilized.

6 Appendix

Proof of Proposition 1. We first establish the following intermediate result.

Lemma 1 *If there does not exist an alternative x such that $P_{xy} \in (0,1)$ and $P_{xz} \in (0,1)$, then $P_{BA}P_{CB}P_{AC} + P_{AB}P_{BC}P_{CA} = 0$.*

Proof. Given the stated hypothesis, there must exist alternatives x and y such that $P_{xy} = 1$. Without loss of generality, suppose $P_{AB} = 1$. It suffices to show that $P_{BC}P_{CA} = 0$. This is trivially true if $P_{CA} = 0$. If $P_{CA} \in (0,1)$, then $P_{BC} \notin (0,1)$ given the stated hypothesis. In addition, if $P_{CA} \in (0,1)$, then $P_{BC} \neq 1$, since $P_{AB} = 1$ and $P_{BC} = 1$ imply that $P_{AC} = 1$, a contradiction. Therefore, $P_{CA} \in (0,1)$ implies that $P_{BC} = 0$. On the other hand, if $P_{CA} = 1$, then this combined with $P_{AB} = 1$ give $P_{BC} = 0$. $\square$

Since $\boldsymbol{\Pi}$ has a unique ergodic set, an invariant distribution of $\boldsymbol{\Pi}$ must be unique. To see that the sequence $\{W_{x,n}\}$ converges uniquely for any x, first note that

$$\boldsymbol{\Pi}^2 = \begin{array}{|c|c|c|}\hline P_{AB}P_{AC} + P_{BA}P_{BC} & P_{BA}P_{CB} & P_{AB}P_{CA} \\ \hline P_{CA}P_{BC} & P_{AB}P_{AC} + P_{CA}P_{CB} & P_{AC}P_{BA} \\ \hline P_{AC}P_{CB} & P_{AB}P_{BC} & P_{BC}P_{BA} + P_{CB}P_{CA} \\ \hline \end{array}$$

and

$$\boldsymbol{\Pi}^3 = \begin{array}{|c|c|c|}\hline Q & P_{AB}(1-Q) & P_{BA}(1-Q) \\ \hline P_{AC}(1-Q) & Q & P_{CA}(1-Q) \\ \hline P_{BC}(1-Q) & P_{CB}(1-Q) & Q \\ \hline \end{array},$$

where $Q = P_{BA}P_{CB}P_{AC} + P_{AB}P_{BC}P_{CA}$. There are two cases to consider.

- There exists at least one alternative x such that $P_{xy} \in (0,1)$ and $P_{xz} \in (0,1)$. In this case, at least one column of $\boldsymbol{\Pi}^2$ has entries that are all strictly positive. Therefore, by Theorem 11.4 of Stokey and Lucas (1989), the sequence $\{\boldsymbol{\Theta\Pi}^{n-1}\}$ converges to the unique invariant distribution of $\boldsymbol{\Pi}$.
- There does not exist an alternative x such that $P_{xy} \in (0,1)$ and $P_{xz} \in (0,1)$. In this case, $Q = 0$ by Lemma 1. Therefore, $\boldsymbol{\Pi}^{2n-1} = \boldsymbol{\Pi}$ and $\boldsymbol{\Pi}^{2n} = \boldsymbol{\Pi}^2$ for all $n \geq 1$. In addition, there must exist alternatives x and y such that $P_{xy} = 1$. Without loss of generality, suppose $P_{AB} = 1$, which, as shown in the proof of Lemma 2, implies that $P_{BC} = 0$ or $P_{CA} = 0$. These conditions yield, for all $n > 1$, $W_{B,n} = \boldsymbol{\Theta\Pi}\mathbf{P}_B = \boldsymbol{\Theta\Pi}^2\mathbf{P}_B = 0$ and

$$W_{A,n} = \boldsymbol{\Theta\Pi}\mathbf{P}_A = \boldsymbol{\Theta\Pi}^2\mathbf{P}_A = \begin{cases} 1 & \text{if } P_{CA} = 0 \\ P_{AC} & \text{if } P_{BC} = 0 \end{cases}.$$

In addition, we have from direct calculation $\boldsymbol{\Theta}^* = \left[\, \frac{1}{2}\ \frac{1}{2}\ 0 \,\right]$ if $P_{CA} = 0$ and $\boldsymbol{\Theta}^* = \left[\, \frac{P_{AC}}{2}\ \frac{1}{2}\ \frac{P_{CA}}{2} \,\right]$ if $P_{BC} = 0$, which give $\boldsymbol{\Theta}^*\mathbf{P}_B = 0$ and

$$\boldsymbol{\Theta}^*\mathbf{P}_A = \begin{cases} 1 & \text{if } P_{CA} = 0 \\ P_{AC} & \text{if } P_{BC} = 0 \end{cases}.$$

Therefore, given any alternative x, $W_{x,n} = \boldsymbol{\Theta}^*\mathbf{P}_x$ for all $n > 1$. □

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