Physics 742 – Graduate Quantum Mechanics 2

First Exam, Spring 2024

The value of each question is listed in square brackets at the start of the problem or part.

1. [20] A spin-½ particle is in the spin state $\left|\psi_\phi\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ e^{i\phi}\end{pmatrix}$, but the phase ϕ is randomly chosen in the range $0 < \phi < \pi$. Note that the Pauli matrices σ_i are given at the end of the test.
   (a) [7] What is the state operator ρ? Check that the trace has the correct value.
   (b) [7] What would be the expectation value of each of the three spin operators $S_i = \frac{1}{2}\hbar\sigma_i$?
   (c) [6] True or false: If the Hamiltonian is $H = \omega S_y$, the state operator is independent of time.

2. [20] This problem should be worked entirely in terms of the Heisenberg formulation. A particle of mass m in one dimension has Hamiltonian $H = P^2/2m - FX$.
   (a) [7] Find in this formalism equations for the time derivatives $\frac{d}{dt}X(t)$ and $\frac{d}{dt}P(t)$.
   (b) [6] Find $P(t)$ in terms of $P(0)$ and t. Then find $X(t)$ in terms of $X(0)$, $P(0)$ and t.
   (c) [7] Show that there is a lower limit on the uncertainties $[\Delta x(t)][\Delta x(0)] \geq Ct$, and find C.

3. [15] A particle of mass m in one dimension is in the potential $V(x) = \begin{cases} -\alpha x & x < 0, \\ \beta x & x > 0. \end{cases}$
   Using the WKB method, estimate the energy of the n'th eigenstate.

4. [15] A particle of mass m lies in the potential $V(x) = B|x|^3$. Estimate the energy of the ground state energy by the variational method using the trial wave function $\psi(x) = e^{-\alpha x^2/2}$.

5. [15] A particle of mass m lies in the 2D infinite square square well with allowed region $0 < x < a$ and $0 < y < a$. In addition, there is a small perturbation of the form $W(x,y) = \lambda\cos(\pi x/a)\cos(\pi y/a)$, where λ is small.
   (a) [2] What are the exact eigenstates and energies in the limit of no perturbation, $\lambda = 0$?
   (b) [13] Find the ground state wave function to first order and the energy to second order in λ.

6. [15] An electron is trapped in a Coulomb potential of the form $V_C(\mathbf{r}) = Ar$. It is in one of the $l = 1$ states, so its space wave function looks like $\psi(\mathbf{r}) = R(r)Y_1^m(\theta,\phi)$ where $Y_1^m(\theta,\phi)$ is a spherical harmonic and $R(r)$ is a radial wave function.
   (a) [8] Given that $l = 1$, what are the possible values of j, the total angular momentum quantum number? For each of these values of j, work out the corresponding eigenvalue of $\mathbf{L}\cdot\mathbf{S}$.
   (b) [7] Find the energy splitting $\Delta\varepsilon'$ between the different states you found in part (a) due to spin-orbit coupling. Since you don't know the radial wave function, you will have to leave one integral undone.

Possibly Helpful Formulas:

State Operators

$$i\hbar\frac{d}{dt}\rho = [H,\rho]$$

Pauli Matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Heisenberg Picture

$$\frac{d}{dt}A(t) = \frac{i}{\hbar}\left[H(t), A(t)\right]$$

Spin-Orbit Coupling

$$W_{\mathrm{SO}} = \frac{g}{4m^2c^2}\frac{1}{r}\frac{dV_c(r)}{dr}\mathbf{L}\cdot\mathbf{S}$$

1D Infinite Square Well

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right)$$

$$E_n = \frac{\pi^2\hbar^2 n^2}{2ma^2}$$

Generalized Uncertainty

$$(\Delta a)(\Delta b) \ge \tfrac{1}{2}\left|\left\langle i[A,B]\right\rangle\right|$$

Possibly Helpful Integrals:
In the equations below A is positive, and in the last equation n, m and p are positive integers.

$$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1}$$

$$\int_0^\infty x^n e^{-Ax^2}\,dx = \frac{1}{2A^{(n+1)/2}}\Gamma\left(\frac{n+1}{2}\right), \quad \Gamma(1)=\Gamma(2)=1, \quad \Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}, \quad \Gamma\left(\tfrac{3}{2}\right)=\tfrac{1}{2}\sqrt{\pi}.$$

$$\int_0^a \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi m x}{a}\right)\cos\left(\frac{\pi p x}{a}\right)dx = \frac{a}{4}\left(\delta_{n,m+p} + \delta_{m,n+p} - \delta_{p,n+m}\right).$$

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